Firm Dynamics without Free-Entry

741 Macroeconomics
Topic 4

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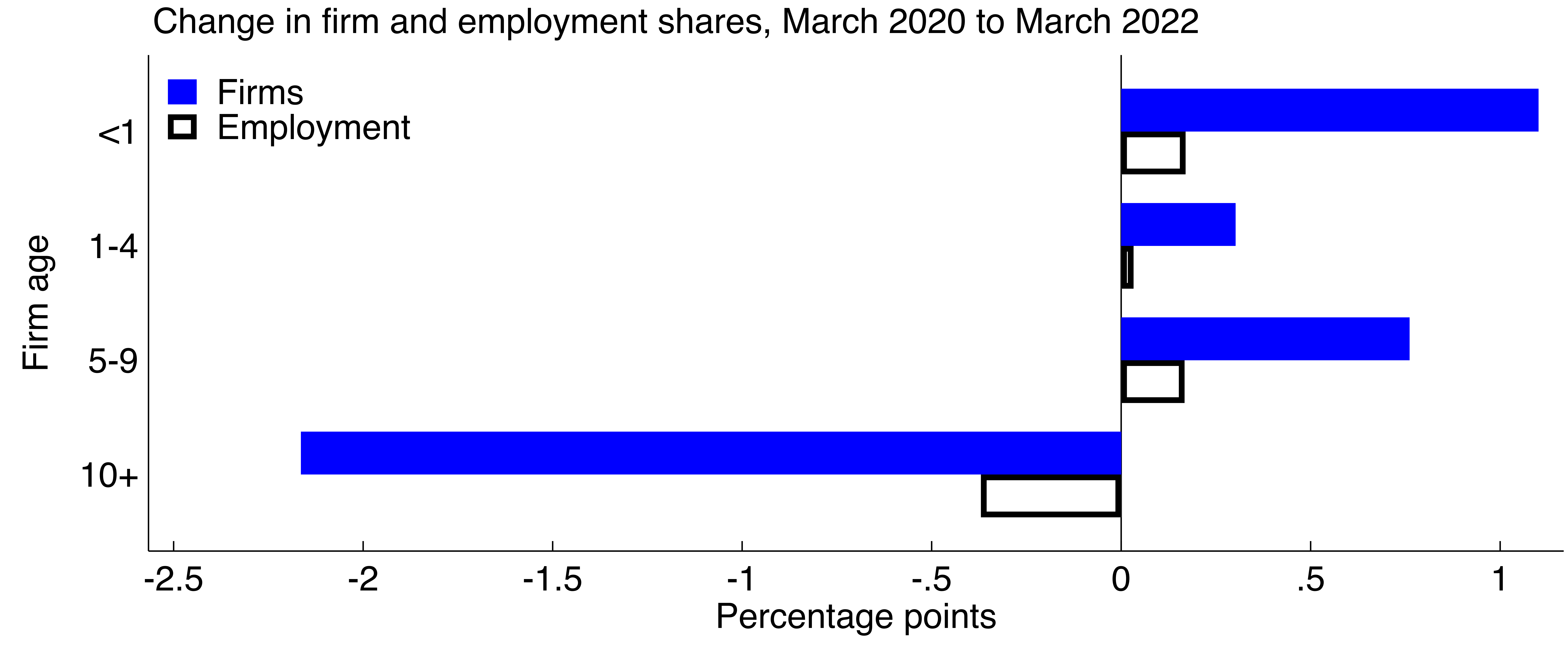
Exit and Entry During COVID-19

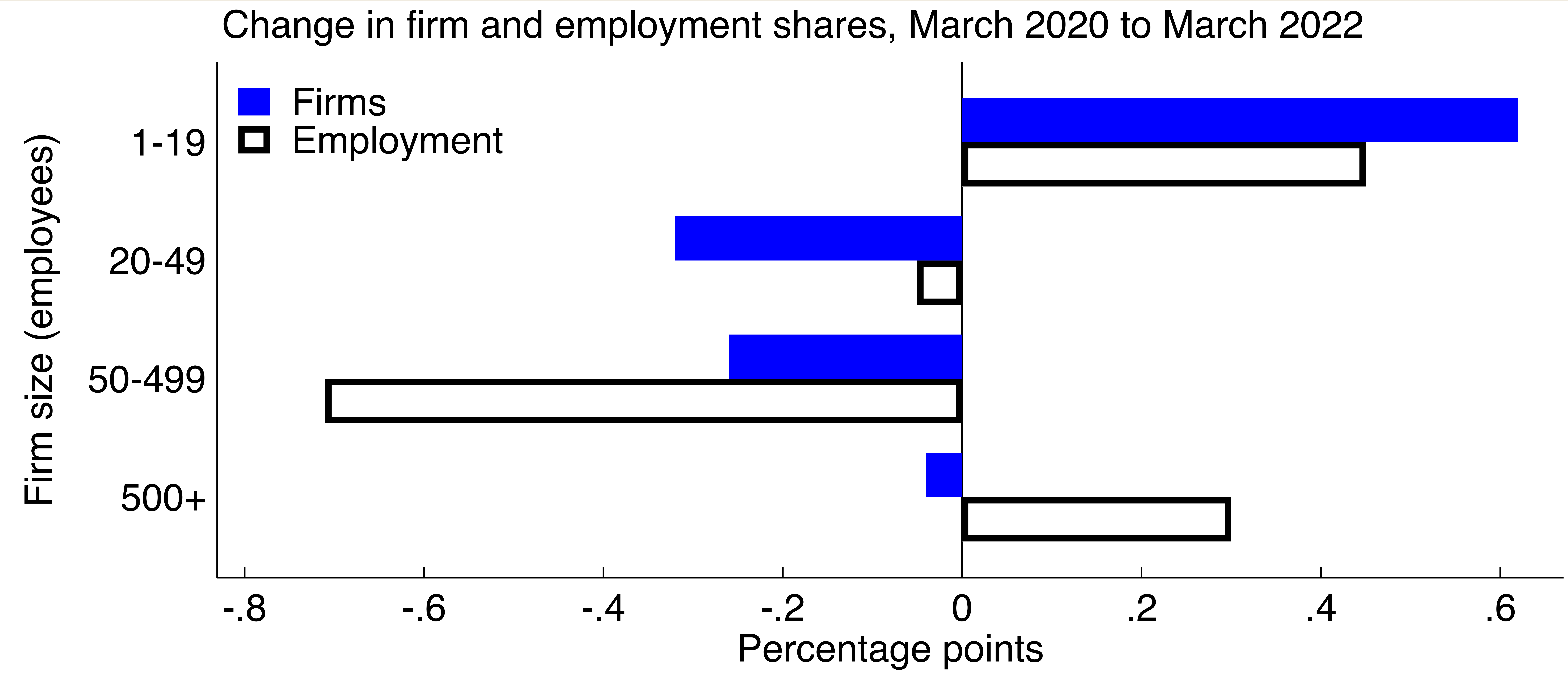
Source: Decker and Haltiwanger (2024)

Now Firms are Younger



Source: Decker and Haltiwanger (2024)

Now Firms are Smaller

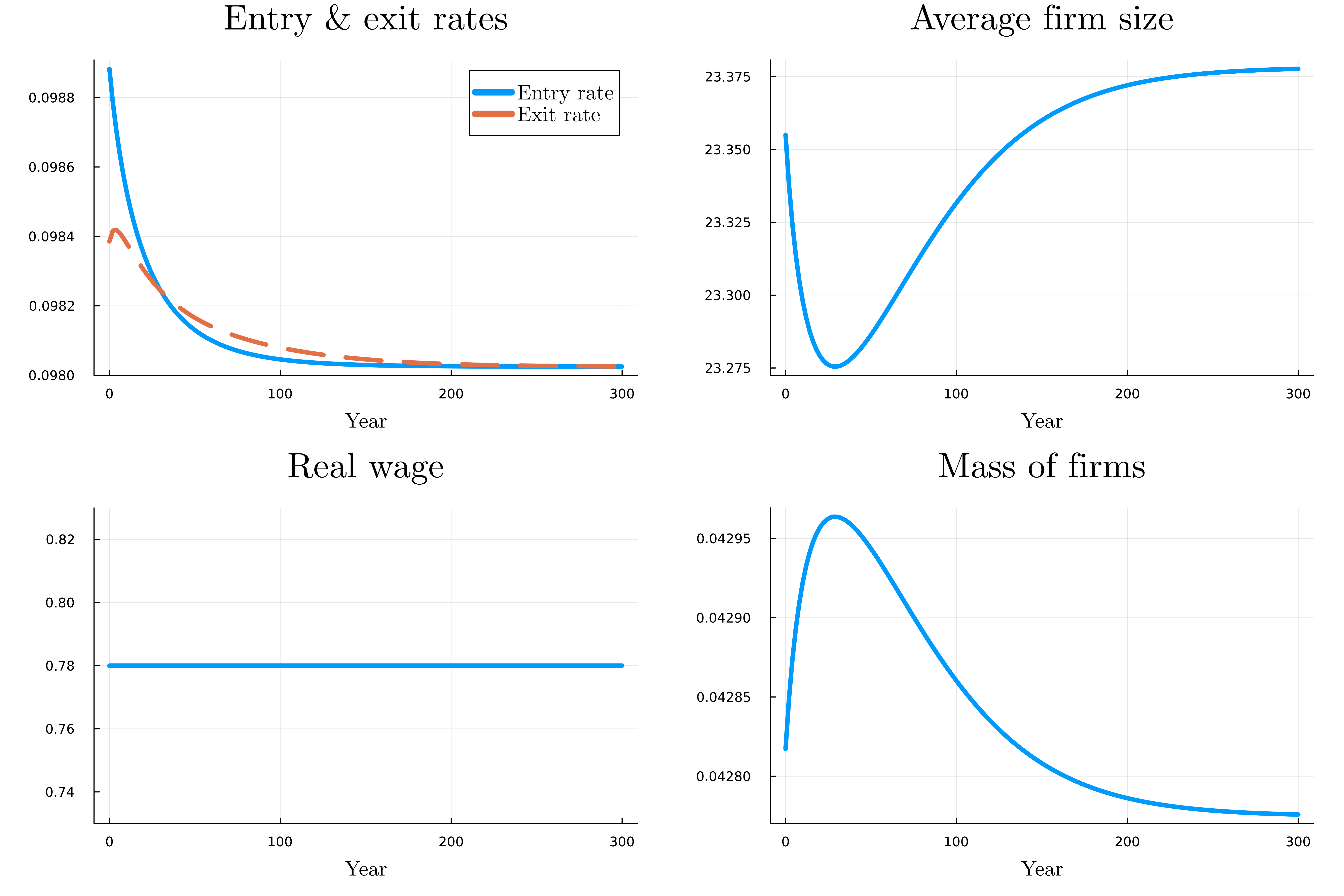


Source: Decker and Haltiwanger (2024)

Firm Destruction Shock in Hopenhayn-Rogerson

- COVID-19 induced a spike in firm exits
- With a slight lag, there was a surge in firm entry
- Reflecting these exit and entry dynamics, firms are now younger and smaller
- With an economy dominated by smaller firms, is the labor demand weaker?
- Suppose we feed firm destruction shocks in our model, what happens?

Response to Firm Exit Shock



How Free is Free-Entry?

- How free is free-entry? Is entry infinitely elastic to entry value?
- No existing estimates (if you estimate it, that will be a great paper)
- Why should we relax free-entry assumption then?

1. Free-entry is not necessarily a benchmark assumption
    - Some firm dynamics models abstract from entry & exits
    - Is this an innocuous assumption or not?
2. There is a general lesson in studying how to solve a model without free-entry
    - This is a class of model where distribution matters for macro!
3. Hard to believe entry is infinitely elastic, especially in the short-run

Hopenhayn-Rogerson without Free-Entry

Relaxing Free-Entry

- We assume that the mass of potential firms is finite at $M \times L_t$
- Potential firms draw entry costs from the distribution $H(c^e)$ iid across time/firms
- Let $\hat{c}^e$ be the cut-off such that potential firms are indifferent to enter or not:

$$\int v_t(z)\psi(z)dz = \hat{c}_e$$

  - Potential firms with $c^e \leq \hat{c}^e$ enter
  - Potential firms with $c^e > \hat{c}^e$ do not enter
- The mass of entrants is

$$m_t = M \times L_t \times H(\hat{c}_t^e)$$

Micro-Founding Inelastic Entry

- Suppose that $1/c^e$ follows Pareto so that

$$\text{Prob}(1/\tilde{c}^e \leq 1/c^e) = 1 - (1/\bar{c}^e)^{\nu}(1/c^e)^{-\nu}$$

$$\Leftrightarrow \quad H(c^e) \equiv \text{Prob}(\tilde{c}^e \leq c^e) = ((1/\bar{c}^e)c^e)^{\nu}$$

- The mass of entry is now

$$m_t = M \times L_t \times \left(\frac{1}{\bar{c}^e} \int v_t(z)\psi(z)dz \right)^{\nu}$$

  - As $\nu \to \infty$, we recover the case of free entry: $\int v_t(z)\psi(z)dz = \bar{c}^e$
  - As $\nu \to 0$, the mass of entry per capita is fixed: $m_t/L_t = M$
  - More generally, ν governs the elasticity of entry w.r.t. firm value

Equilibrium System

$$\min\left\{ rv_t(z) - \pi(z;w_t) - \mu(z)v_t'(z) - \frac{1}{2}\sigma(z)^2 v_t''(z) - \partial_t v_t(z), v_t(z) - \underline{v} \right\} = 0$$

$$v\left(\underline{z}_t\right) = \underline{v}$$

$$m_t = M \times L_t \times \left(\frac{1}{\bar{c}^e} \int v_t(z)\psi(z)dz \right)^{\nu}$$

$$\partial_t g_t(z) = -\partial_z[\mu(z)g_t(z)] + \frac{1}{2}\partial_{zz}^2\left[\sigma(z)^2 g_t(z)\right] + m_t\psi(z) \quad \text{for } z > \underline{z}_t$$

$$\int n(z;w_t)g_t(z)dz = L_t$$

- Now we lost the block recursive property
- Entry alone does *not* pin down wages. The whole distribution matters!

Normalized Equilibrium Conditions

- Define $\tilde{m}_t \equiv m_t / L_t$ and $\tilde{g}_t(z) \equiv g_t(z)/L_t$

$$\min \left\{ r v_t(z) - \pi(z; w_t) - \mu(z) v_t'(z) - \frac{1}{2}\sigma(z)^2 v_t''(z) - \partial_t v_t(z),\, v_t(z) - \underline{v} \right\} = 0$$

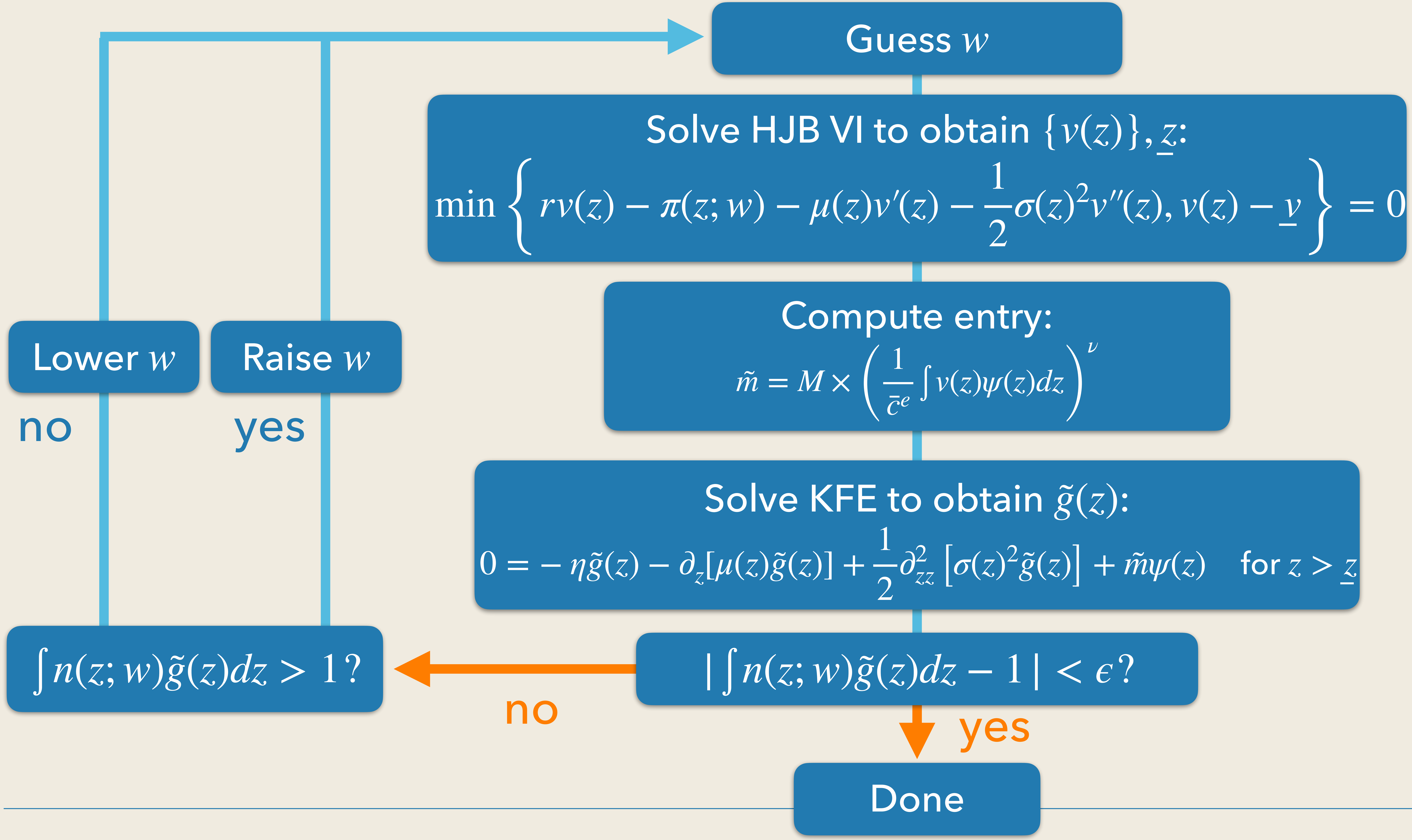
$$v\left(\underline{z}_t\right) = \underline{v}$$

$$\tilde{m}_t = M \times \left(\frac{1}{\bar{c}^e} \int v_t(z) \psi(z) dz \right)^{\nu}$$

$$\partial_t \tilde{g}_t(z) = -\eta \tilde{g}_t(z) - \partial_z[\mu(z) \tilde{g}_t(z)] + \frac{1}{2}\partial_{zz}^2 \left[\sigma(z)^2 \tilde{g}_t(z)\right] + \tilde{m}_t \psi(z) \quad \text{for } z > \underline{z}_t$$

$$\int n(z; w_t) \tilde{g}_t(z) dz = 1$$

Steady State Algorithm



Solving Transition Dynamics using Sequence Space Jacobians

Goal

- Suppose the economy is initially in a steady state at $t = 0$
- After $t = 0$, the population growth changes over time $\{\eta_t\}$
- How do we simulate the transition dynamics?

Equilibrium System

$$\min\left\{rv_t(z)-\pi(z;w_t)-\mu(z)v_t'(z)-\frac{1}{2}\sigma(z)^2v_t''(z)-\partial_t v_t(z),\, v_t(z)-\underline{v}\right\}=0 \qquad \text{(HJB-VI)}$$

$$v\left(\underline{z}_t\right)=\underline{v} \qquad \text{(Exit)}$$

$$\tilde{m}_t = M\times\left(\frac{1}{\bar{c}^e}\int v_t(z)\psi(z)dz\right)^{\nu} \qquad \text{(Entry)}$$

$$\partial_t\tilde{g}_t(z) = -\eta_t\tilde{g}_t(z)-\partial_z[\mu(z)\tilde{g}_t(z)]+\frac{1}{2}\partial^2_{zz}\left[\sigma(z)^2\tilde{g}_t(z)\right]+\tilde{m}_t\psi(z) \quad \text{for } z>\underline{z}_t \qquad \text{(KFE)}$$

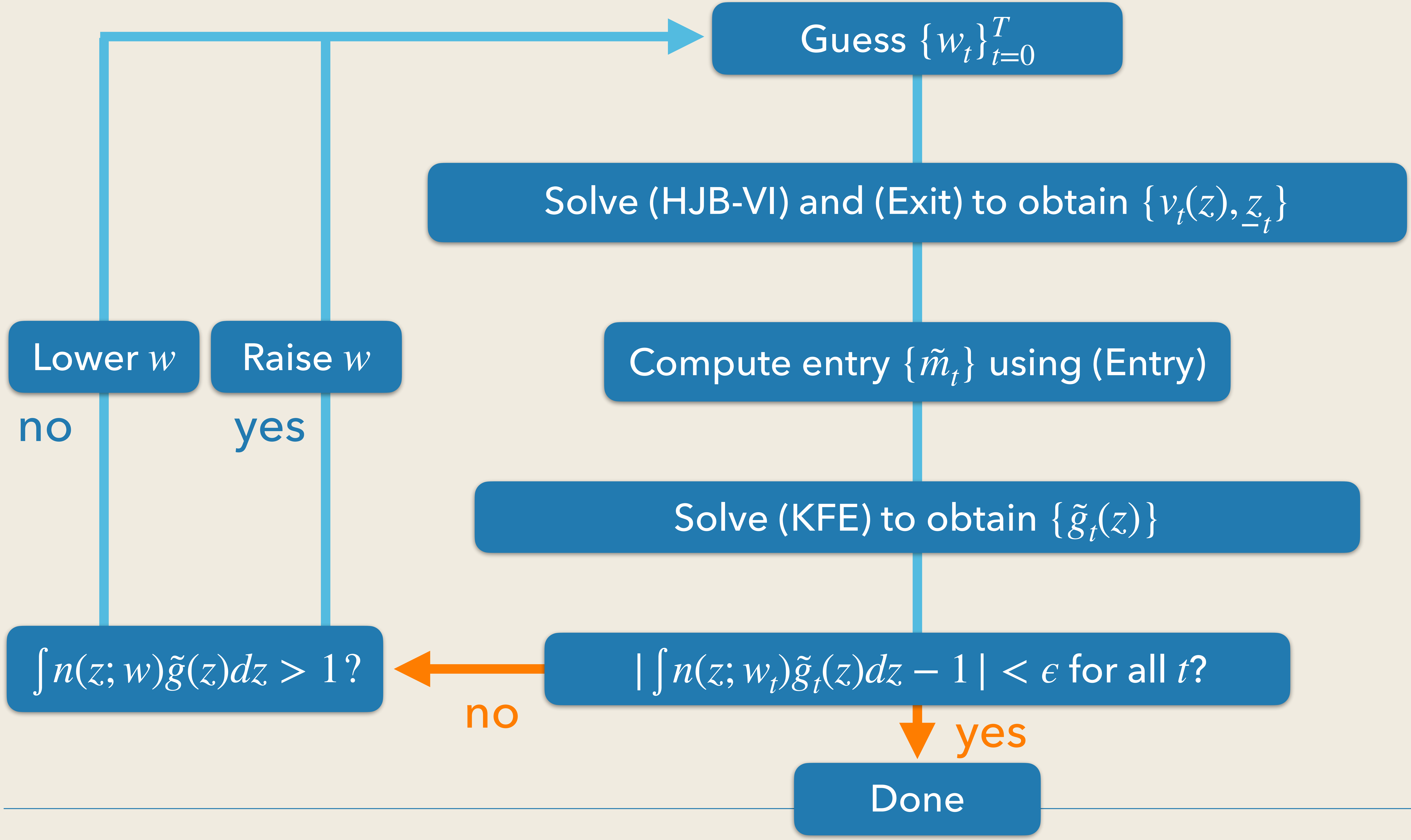
$$\int n(z;w_t)\tilde{g}_t(z)dz=1 \qquad \text{(MC)}$$

- We need to find a sequence of wages $\{w_t\}_{t=0}^{\infty}$ that clear the labor market
- It is useful to start from a "naive" algorithm

Algorithm 10 Years Ago

- Assume that, at $t = T$, the economy is in a steady state
  - Make the problem finite-dimensional
- The "naive" algorithm is to keep guessing $\{w_t\}_{t=0}^{T}$ until labor markets clear for all t

"Naive" Algorithm 10 Years Ago




Wage, $\{w_t\}$
Shock, $\{\eta_t\}$
HJB, $\{v_t(z)\}$
Exit, $\{\underline{z}_t\}$
Entry, $\{\tilde{m}_t\}$
Labor demand, $\{n_t(z)\}$
KFE, $\{\tilde{g}_t(z)\}$
Labor Market, $\mathcal{N}(\{w_t\},\{\eta_t\}) \equiv \int n_t(z)\tilde{g}_t(z)dz = 1$

Equilibrium System

$$\mathcal{N}_t\left(\mathbf{w},\boldsymbol{\eta}\right) = 1$$

- We look for first-order approximation around the steady state
- Why?
  - Instantaneous to obtain a solution, as we will see
  - Often cases in practice, there is little non-linearity
  - It can be the basis for solving non-linear solutions

Linearized Solution

- Discretize time with S grid points, and let $\Delta t \equiv T/S$ be the time-interval
- First-order solution:

$$\mathcal{N}_w d\boldsymbol{w} + \mathcal{N}_\eta d\boldsymbol{\eta} = 0$$

where $\mathcal{N}_w \equiv \left[\frac{\partial \mathcal{N}_t}{\partial w_s}\right]_{t,s}$ and $\mathcal{N}_\eta \equiv \left[\frac{\partial \mathcal{N}_t}{\partial \eta_s}\right]_{t,s}$ are $S \times S$ Jacobian matrix

- Solving for $d\boldsymbol{w}$,

$$d\boldsymbol{w} = -\left(\mathcal{N}_w\right)^{-1} \mathcal{N}_\eta d\boldsymbol{\eta}$$

Obtaining Sequence Space Jacobians

- How do we obtain the sequence-space Jacobian, $[\mathcal{N}_w]_{t,s}$?
  - Changes in labor demand at time t in response to changes in w at time s
- Again, let us think through a "naive" algorithm
  1. Consider $\boldsymbol{w}' \equiv [w^{ss} + dw_0, w^{ss}, \ldots, w^{ss}]$
  2. Given $\boldsymbol{w}'$, solve HJBVI backward to obtain $\tilde{\boldsymbol{m}}', \underline{z}', \{\boldsymbol{n}(z)'\}$
  3. Use $\tilde{\boldsymbol{m}}', \underline{z}'$ to solve KFE forward to obtain $\{\tilde{\boldsymbol{g}}(z)'\}$
  4. Use $\{\boldsymbol{n}(z)'\}$ and $\{\tilde{\boldsymbol{g}}(z)'\}$ to compute $\mathcal{N}'$ and thereby $[\mathcal{N}_w]_{t,0} \equiv \dfrac{\partial \mathcal{N}_t}{\partial w_s} = \dfrac{\mathcal{N}_t' - \mathcal{N}^{ss}}{dw_0}$
  5. Repeat this with dw_s for all $s = \Delta t, 2\Delta t, \ldots, S\Delta t$
- This is very time-consuming! Need S backward and S forward iterations
- Can we do better? – Yes, a lot better (Auclert, Bardóczy, Rognlie, Straub, 2021)

Only One Backward Iteration is Needed

- The first key insight:

$$\frac{d\underline{z}_t}{dw_s} = \begin{cases} 0 & s \leq t \\ \frac{d\underline{z}_{T-(s-t)}}{dw_T} & s > t \end{cases}, \qquad \frac{d\tilde{m}_t}{dw_s} = \begin{cases} 0 & s \leq t \\ \frac{d\tilde{m}_{T-(s-t)}}{dw_T} & s > t \end{cases}$$

  HJB-VI is (i) forward-looking and (ii) timeless:

  (i) shock that happened in the past is irrelevant to my policy functions

  (ii) I care about the distance to the future shock, not the calendar time

- $\frac{d\underline{z}_t}{dw_s}$ and $\frac{d\tilde{m}_t}{dw_s}$ can be obtained from a single backward iteration in response to dw_T
- With $n_t(z) = (\alpha / w_t)^{\frac{1}{1-\alpha}} z$, $\frac{d\boldsymbol{n}_t}{dw_s}$ is trivial to obtain
- Reduce computational time by a factor of S

Matrix Notation

- We write the KFE in a matrix form as

$$\frac{\boldsymbol{g}_t - \boldsymbol{g}_{t-\Delta t}}{\Delta t} = [\tilde{\boldsymbol{A}}_t]' \boldsymbol{g}_t + \tilde{\boldsymbol{\psi}}$$

$$\Leftrightarrow \qquad \tilde{\boldsymbol{g}}_t = \underbrace{\left[\boldsymbol{I} - \Delta t \times [\tilde{\boldsymbol{A}}_t]^T\right]^{-1}}_{\equiv \boldsymbol{P}_t} \times \left[\tilde{\boldsymbol{g}}_{t-\Delta t} + \Delta t \times \tilde{\boldsymbol{\psi}}\right]$$

- The labor market clearing is, in a matrix form,

$$\mathcal{N}_t = \boldsymbol{n}_t' \tilde{\boldsymbol{g}}_t$$

where $\boldsymbol{n}_t \equiv [n_t(z)]$ and $\tilde{\boldsymbol{g}}_t \equiv [\tilde{g}_t(z)]$

Response at $t = 0$ to $s = 0$ Shock

- Given $\frac{d\underline{z}_t}{dw_s}$, $\frac{d\tilde{m}_t}{dw_s}$, and $\frac{dn_t(z)}{dw_s}$, we compute

$$[\mathcal{N}_w]_{0,0} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_0}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_0} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_0} = \frac{d\boldsymbol{P}_0}{dw_0} \times \tilde{\boldsymbol{g}}^{ss}$$

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Impact through changes in n holding distribution fixed

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Response at $t = 0$ to $s = 0$ Shock

Impact through changes in distributions

holding n fixed

- Given $\frac{d\underline{z}_t}{dw_s}$, $\frac{d\tilde{m}_t}{dw_s}$, and $\frac{dn_t(z)}{dw_s}$, we compute

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- From this, we can obtain $[\mathcal{N}_w]_{t,0}$ immediately as well because

$$[\mathcal{N}_w]_{t,0} = [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_t}{dw_0}$$

$$\frac{d\tilde{\boldsymbol{g}}_t}{dw_0} = \boldsymbol{P}^{ss} \times \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_0}$$

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$$\frac{d\tilde{\boldsymbol{g}}_t}{dw_0} = \boldsymbol{P}^{ss} \times \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_0}$$

  - After the shock at $t = 0$, policy functions are the same as ones in the steady-state
  - The distribution of transition is governed by steady-state objects $\boldsymbol{P}^{ss}$

Now we know the first column

$$\mathcal{N}_w \equiv \begin{bmatrix} \left[\mathcal{N}_w\right]_{0,0} & \\ \left[\mathcal{N}_w\right]_{\Delta t,0} & \\ \left[\mathcal{N}_w\right]_{2\Delta t,0} & ? \\ \vdots & \\ \vdots & \\ \left[\mathcal{N}_w\right]_{S\times\Delta t,0} & \end{bmatrix}$$

Second Column

- Given $\frac{d\underline{z}_t}{dw_s}$, $\frac{d\tilde{m}_t}{dw_s}$, and $\frac{d\boldsymbol{n}_t}{dw_s}$, we compute

$$[\mathcal{N}_w]_{0,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_{\Delta t}}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} = \frac{d\boldsymbol{P}_0}{dw_{\Delta t}} \times \boldsymbol{g}^{ss}$$

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- What about $[\mathcal{N}_w]_{\Delta t,\Delta t}$?

Second Column

- Given $\frac{d\underline{z}_t}{dw_s}$, $\frac{d\tilde{m}_t}{dw_s}$, and $\frac{d\boldsymbol{n}_t}{dw_s}$, we compute

$$[\mathcal{N}_w]_{0,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_{\Delta t}}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} = \frac{d\boldsymbol{P}_0}{dw_{\Delta t}} \times \boldsymbol{g}^{ss}$$

- What about $[\mathcal{N}_w]_{\Delta t,\Delta t}$?

$$\begin{aligned} [\mathcal{N}_w]_{\Delta t,\Delta t} &\equiv \left[\frac{d\boldsymbol{n}_{\Delta t}}{dw_{\Delta t}}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}} \\ &= \left[\frac{d\boldsymbol{n}_0}{dw_0}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_0} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}} - [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_0} \\ &= [\mathcal{N}_w]_{0,0} + [\boldsymbol{n}^{ss}]' \boldsymbol{P}^{ss} \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} \end{aligned}$$

Second Column

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- What about $[\mathcal{N}_w]_{\Delta t,\Delta t}$?

$$[\mathcal{N}_w]_{\Delta t,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_{\Delta t}}{dw_{\Delta t}}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}}$$

$$[\mathcal{N}_w]_{0,0} \quad = \left[\frac{d\boldsymbol{n}_0}{dw_0}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_0} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}} - [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_0}$$

$$= [\mathcal{N}_w]_{0,0} + [\boldsymbol{n}^{ss}]' \boldsymbol{P}^{ss} \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}}$$

Second Column

- Given $\frac{dz_t}{dw_s}$, $\frac{d\tilde{m}_t}{dw_s}$, and $\frac{d\boldsymbol{n}_t}{dw_s}$, we compute

$$[\mathcal{N}_w]_{0,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_{\Delta t}}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} = \frac{d\boldsymbol{P}_0}{dw_{\Delta t}} \times \boldsymbol{g}^{ss}$$

- What about $[\mathcal{N}_w]_{\Delta t,\Delta t}$?

$$[\mathcal{N}_w]_{\Delta t,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_{\Delta t}}{dw_{\Delta t}}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}}$$

$$= \left[\frac{d\boldsymbol{n}_0}{dw_0}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_0} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_{\Delta t}}{dw_{\Delta t}} - [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_0}$$

$$[\mathcal{N}_w]_{0,0}$$

$$\frac{d\boldsymbol{P}_{\Delta t}}{dw_{\Delta t}}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}} = \frac{d\boldsymbol{P}_0}{dw_0}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}}$$

$$\frac{d\boldsymbol{P}_0}{dw_0}\tilde{\boldsymbol{g}}^{ss}$$

$$= [\mathcal{N}_w]_{0,0} + [\boldsymbol{n}^{ss}]'\boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}}$$

Second Column

- For $t > \Delta t$

$$[\mathcal{N}_w]_{t,\Delta t} \equiv \left[\frac{d\boldsymbol{n}_t}{dw_{\Delta t}}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_t}{dw_{\Delta t}}$$

$$= \left[\frac{d\boldsymbol{n}_{t-\Delta t}}{dw_0}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_0} + [\boldsymbol{n}^{ss}]'\left[\frac{d\tilde{\boldsymbol{g}}_t}{dw_{\Delta t}} - \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_0}\right]$$

$$= [\mathcal{N}_w]_{t-\Delta t,0} + [\boldsymbol{n}^{ss}]'\left[\frac{d\tilde{\boldsymbol{g}}_t}{dw_{\Delta t}} - \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_0}\right]$$

$$= [\mathcal{N}_w]_{t-\Delta t,0} + [\boldsymbol{n}^{ss}]'\left[\frac{d\boldsymbol{P}_t}{dw_{\Delta t}}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_{\Delta t}} - \frac{d\boldsymbol{P}_{t-\Delta t}}{dw_0}\tilde{\boldsymbol{g}}^{ss} - \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_0}\right]$$

$$= [\mathcal{N}_w]_{t-\Delta t,0} + [\boldsymbol{n}^{ss}]'\boldsymbol{P}^{ss}\left[\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_{\Delta t}} - \frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_0}\right]$$

$$= [\mathcal{N}_w]_{t-\Delta t,0} + [\boldsymbol{n}^{ss}]'(\boldsymbol{P}^{ss})^{\frac{t}{\Delta t}}\frac{d\tilde{\boldsymbol{g}}_0}{dw_{\Delta t}}$$

Repeat the above step until time 0

Now we know the first two columns

$$\mathcal{N}_w \equiv \begin{bmatrix} [\mathcal{N}_w]_{0,0} & [\mathcal{N}_w]_{0,\Delta t} & \\ [\mathcal{N}_w]_{\Delta t,0} & [\mathcal{N}_w]_{\Delta t,\Delta t} & \\ [\mathcal{N}_w]_{2\Delta t,0} & [\mathcal{N}_w]_{2\Delta t,\Delta t} & ? \\ \vdots & & \\ \vdots & & \\ [\mathcal{N}_w]_{S\times\Delta t,0} & [\mathcal{N}_w]_{S\times\Delta t,\Delta t} & \end{bmatrix}$$

Recursive Expression for SSJ

- For $t = 0$ and any s,

$$[\mathcal{N}_w]_{0,s} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_s}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_s} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_s} = \frac{d\boldsymbol{P}_0}{dw_s} \times \tilde{\boldsymbol{g}}^{ss}$$

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- For $t > 0$ and any s,

$$\begin{aligned}
[\mathcal{N}_w]_{t,s} - [\mathcal{N}_w]_{t-\Delta t,s-\Delta t} &= [\boldsymbol{n}^{ss}]'\left[\frac{d\tilde{\boldsymbol{g}}_t}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_{s-\Delta t}}\right] \\
&= [\boldsymbol{n}^{ss}]'\left[\frac{d\boldsymbol{P}_t}{dw_s}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\boldsymbol{P}_{t-\Delta t}}{dw_{s-\Delta t}}\tilde{\boldsymbol{g}}^{ss} - \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right] \\
&= [\boldsymbol{n}^{ss}]'\boldsymbol{P}^{ss}\left[\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right] \\
&= [\boldsymbol{n}^{ss}]'(\boldsymbol{P}^{ss})^{\frac{t}{\Delta t}}\frac{d\tilde{\boldsymbol{g}}_0}{dw_s}
\end{aligned}$$

Recursive Expression for SSJ

- For $t = 0$ and any s,

$$[\mathcal{N}_w]_{0,s} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_s}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_s} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_s} = \frac{d\boldsymbol{P}_0}{dw_s} \times \tilde{\boldsymbol{g}}^{ss}$$

- For $t > 0$ and any s,

$$[\mathcal{N}_w]_{t,s} - [\mathcal{N}_w]_{t-\Delta t,s-\Delta t} = [\boldsymbol{n}^{ss}]'\left[\frac{d\tilde{\boldsymbol{g}}_t}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'\left[\frac{d\boldsymbol{P}_t}{dw_s}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\boldsymbol{P}_{t-\Delta t}}{dw_{s-\Delta t}}\tilde{\boldsymbol{g}}^{ss} - \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'\boldsymbol{P}^{ss}\left[\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'(\boldsymbol{P}^{ss})^{\frac{t}{\Delta t}}\frac{d\tilde{\boldsymbol{g}}_0}{dw_s}$$

The difference btwn :

1. Response at t to shock at s
2. Response at $t - \Delta t$ to shock at $s - \Delta t$

Recursive Expression for SSJ

- For $t = 0$ and any s,

$$[\mathcal{N}_w]_{0,s} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_s}\right]'\tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]'\frac{d\tilde{\boldsymbol{g}}_0}{dw_s} \quad \text{and} \quad \frac{d\tilde{\boldsymbol{g}}_0}{dw_s} = \frac{d\boldsymbol{P}_0}{dw_s} \times \tilde{\boldsymbol{g}}^{ss}$$

- For $t > 0$ and any s,

$$[\mathcal{N}_w]_{t,s} - [\mathcal{N}_w]_{t-\Delta t,s-\Delta t} = [\boldsymbol{n}^{ss}]'\left[\frac{d\tilde{\boldsymbol{g}}_t}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'\left[\frac{d\boldsymbol{P}_t}{dw_s}\tilde{\boldsymbol{g}}^{ss} + \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\boldsymbol{P}_{t-\Delta t}}{dw_{s-\Delta t}}\tilde{\boldsymbol{g}}^{ss} - \boldsymbol{P}^{ss}\frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'\boldsymbol{P}^{ss}\left[\frac{d\tilde{\boldsymbol{g}}_{t-\Delta t}}{dw_s} - \frac{d\tilde{\boldsymbol{g}}_{t-2\Delta t}}{dw_{s-\Delta t}}\right]$$

$$= [\boldsymbol{n}^{ss}]'(\boldsymbol{P}^{ss})^{\frac{t}{\Delta t}}\frac{d\tilde{\boldsymbol{g}}_0}{dw_s}$$

The difference btwn :

1. Response at t to shock at s
2. Response at $t - \Delta t$ to shock at $s - \Delta t$

Firms had one more period in advance to prepare for the shock

Sequence Space Jacobian Algorithm

1. Solve HJB-VI backward in response to a shock at the terminal period, dw_T
   - This gives $\{\frac{d\boldsymbol{n}_t}{dw_s}, \frac{d\underline{z}_t}{dw_s}, \frac{dm_t}{dw_s}\}$ for any t, s because $\frac{dx_t}{dw_s} = \frac{dx_T}{dw_{T-(t-s)}}$

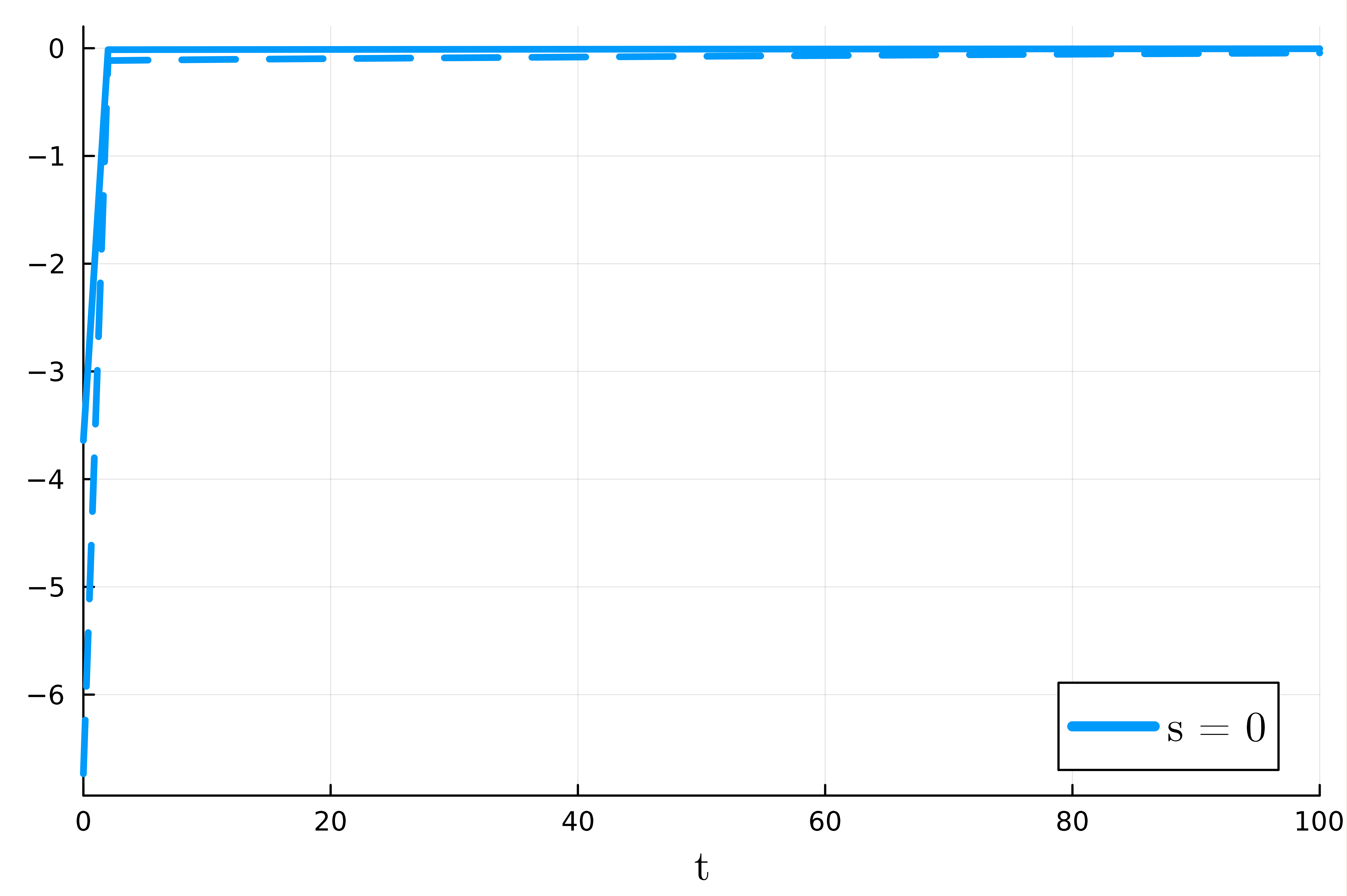
2. For each $s = 0, \Delta t, \ldots, S\Delta t$
   - Compute $[\mathcal{N}_w]_{0,s} \equiv \left[\frac{d\boldsymbol{n}_0}{dw_s}\right]' \tilde{\boldsymbol{g}}^{ss} + [\boldsymbol{n}^{ss}]' \frac{d\tilde{\boldsymbol{g}}_0}{dw_s}$ and $\frac{d\tilde{\boldsymbol{g}}_0}{dw_s} = \frac{d\boldsymbol{P}_0}{dw_s} \times \tilde{\boldsymbol{g}}^{ss}$
   - For each $t = 0, \Delta t, \ldots, S\Delta t$, compute $[\mathcal{N}_w]_{t,s}$ recursively using

$$[\mathcal{N}_w]_{t,s} = [\mathcal{N}_w]_{t-\Delta t, s-\Delta t} + [\boldsymbol{n}^{ss}]' (\boldsymbol{P}^{ss})^{\frac{t}{\Delta t}} \frac{d\tilde{\boldsymbol{g}}_0}{dw_s}$$

⚠ For $t < 0$ or $s < 0$, set $[\mathcal{N}_w]_{t,s} = 0$

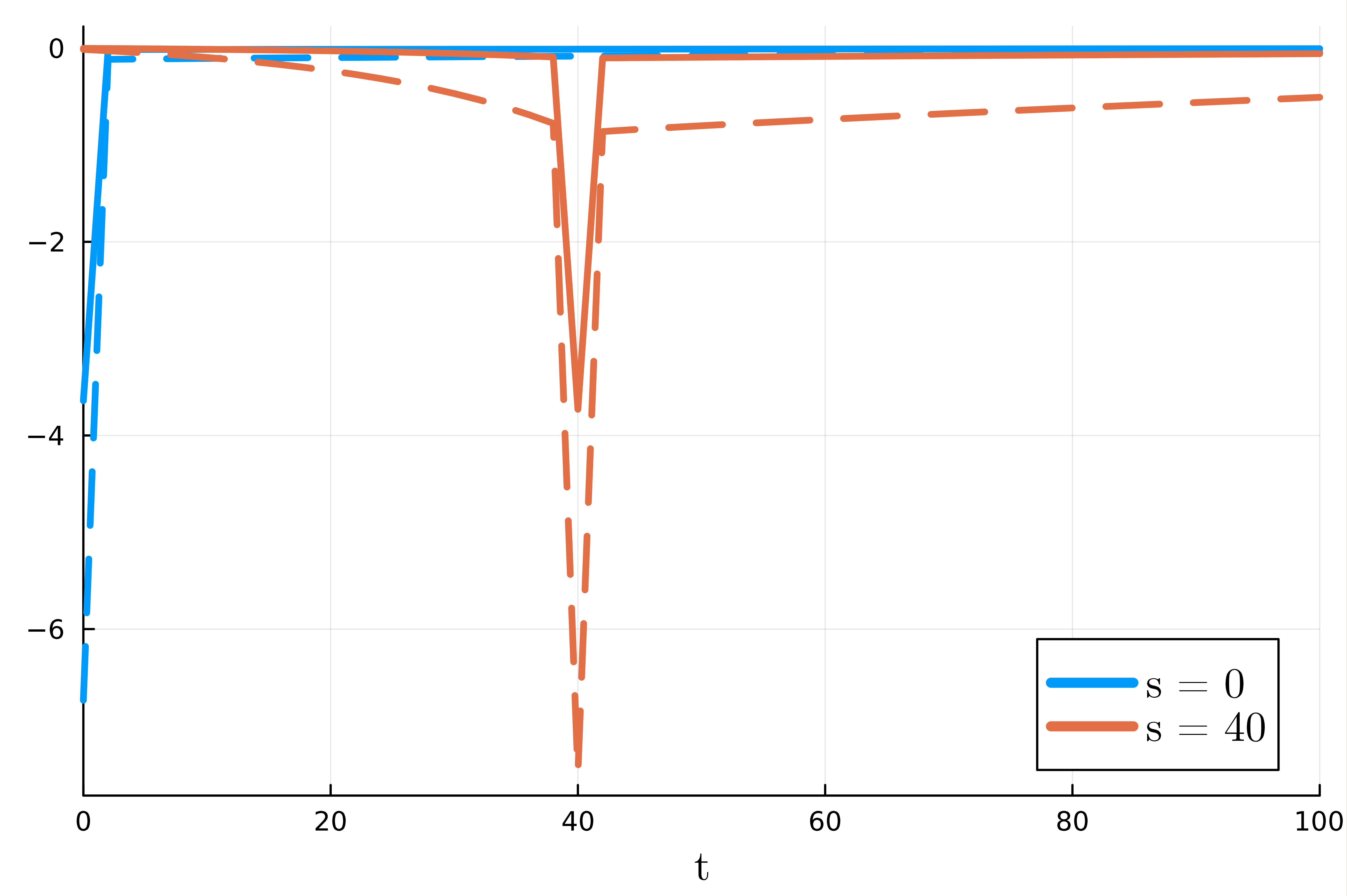
Sequence Space Jacobians: Application

Elements of Jacobian $[\mathcal{N}_w]_{t,s}$



- Solid:
  low entry elasticity, ν
- Dashed:
  high entry elasticity, ν

Elements of Jacobian $[\mathcal{N}_w]_{t,s}$



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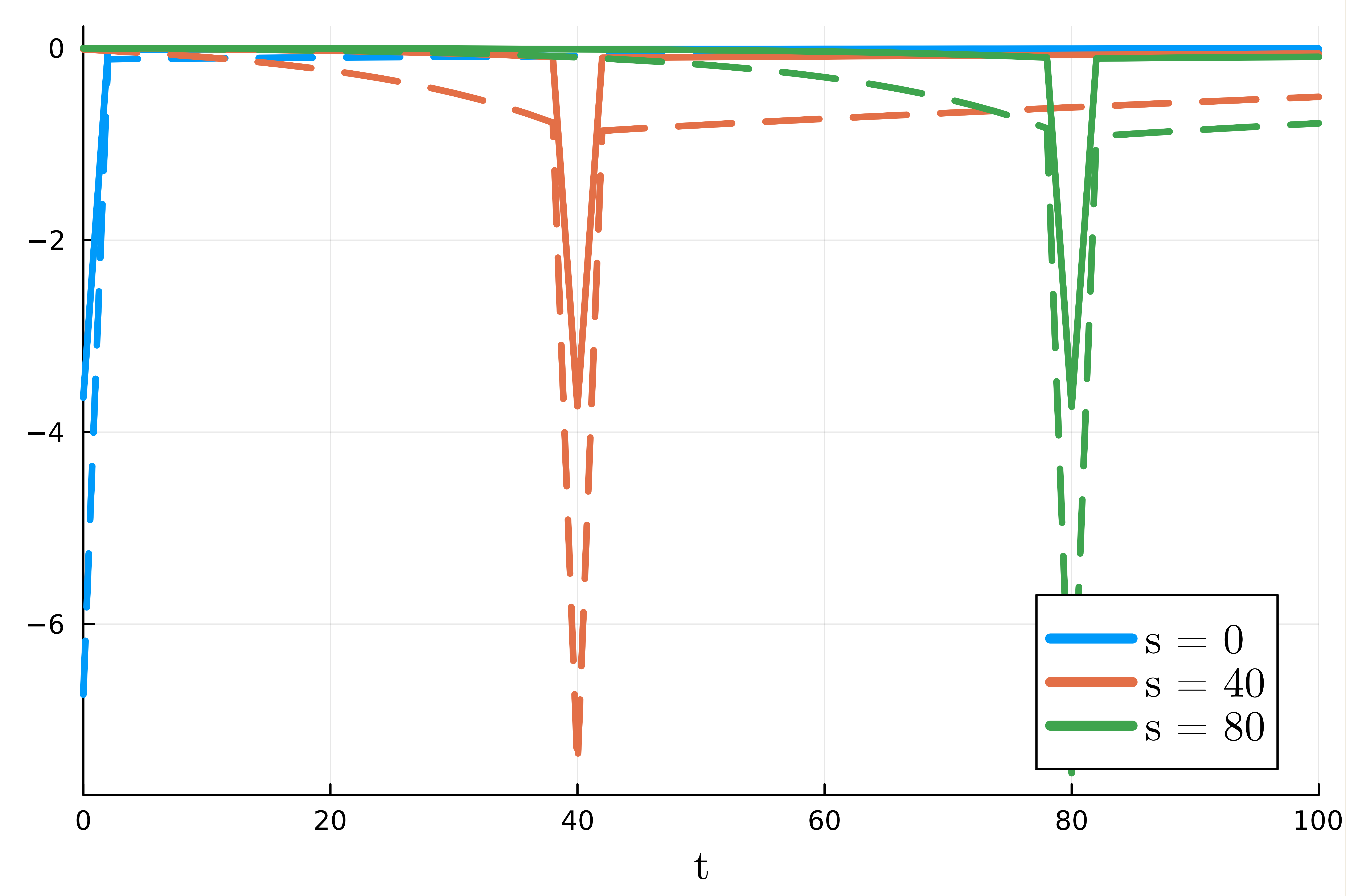
Elements of Jacobian $[\mathcal{N}_w]_{t,s}$



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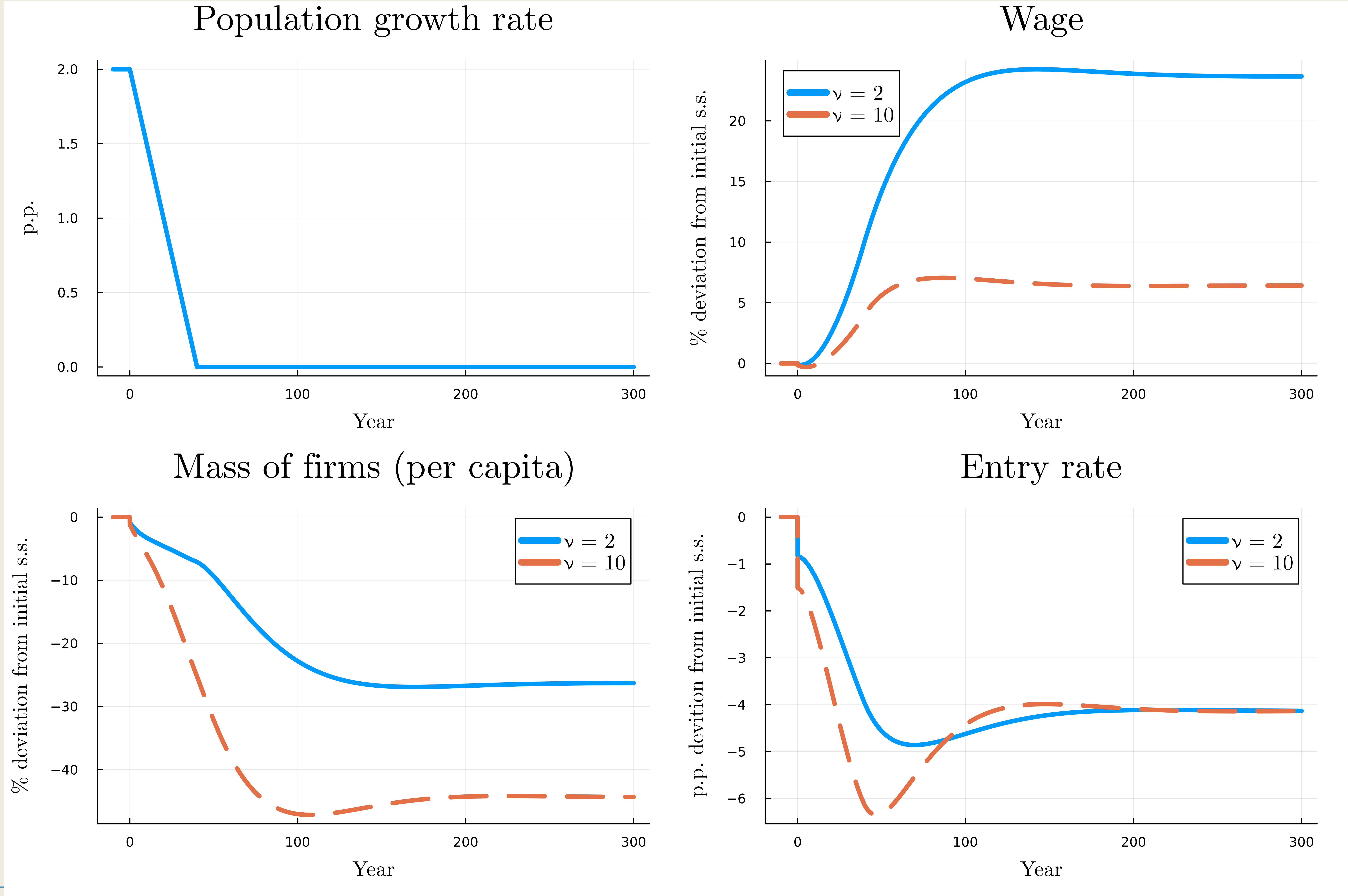
Demographic Origins of Startup Deficits Revisited



Firm Exit Shocks Revisited