The Nature of Labor Reallocation

Masao Fukui

741 Macroeconomics Topic 5

Fall 2024





Firm Employment is Log-Linear in TFP

In Hopenhayn-Rogerson, firm-level employment is given by

n = (z

 $\Leftrightarrow \log n = \frac{1}{1}$

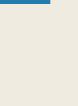
Two implications:

- 1. The elasticity of firm employment w.r.t. (firm-level) TFP shock is above 1
- 2. The elasticity is symmetric to positive & negative shocks
- Is this true in the data?

$$z^{1-\alpha} \alpha/w)^{\frac{1}{1-\alpha}}$$

$$\equiv Z$$

$$\frac{1}{-\alpha} \log Z + const$$





llut, Kehrig & Schneider (2018)

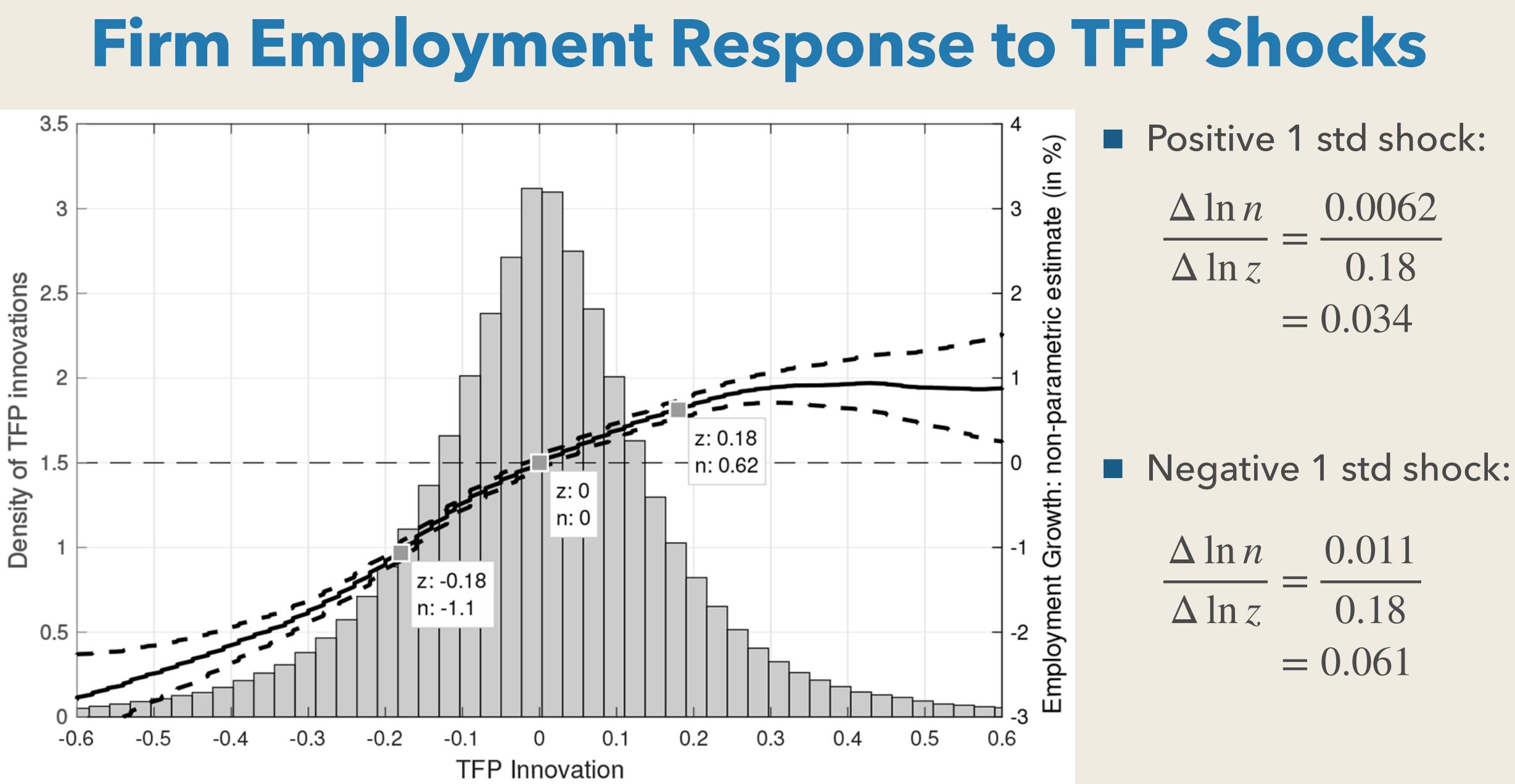
- Focus on US manufacturing establishments (Census data)
- Construct firm-level TFP using Solow residual:

$$\log sr_{it} = \log y_{it} - (\beta_n 1)$$

- Construct firm-level TFP shocks, Z_{it}, assuming $\log sr_{it} = g \times t + \alpha^{l} + \log Z_{it}$
- Q: How does firm-level employment respond to TFP shocks? $\Delta \log n_{it} = h(\Delta \log Z_{it}) + \gamma' X_{it} + \epsilon_{it}$

- $\log n_{it} + \beta_k \log k_{it} + \beta_m \log m_{it})$







In the data,

- 1. The elasticity of firm employment to TFP shock is far below 1
- 2. Elasticity is two times larger for negative shocks than positive shocks



Hopenhayn-Rogerson with Labor Adjustment Costs



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The simplest explanation:

- it is costly to hire workers
- less so to fire workers

Slow to Hire, Quick to Fire



Labor Adjustment Cost

Suppose that firms face

- no cost from firing workers: $\Phi(h, n) = 0$ for $h \le 0$

 \Rightarrow convex cost in hiring & free firing

- The firm employment evolves $dn_t = hdt$
- Firms never want to jump n upward
- But firms may jump n downward

• a flow adjustment cost in hiring $h \ge 0$ of the form $\Phi(h, n)$ with $\partial_h \Phi > 0, \partial_{hh}^2 \Phi > 0$

• Why? – The cost of doing so is $\lim_{dt\to 0} \Phi(h, n) dt = \infty$ with $h = \frac{n' - n}{dt}$ and n' > n





- The rest of the model remains the same as before
- The production function is

- and firms incur a fixed operating cost c_f
- Firm's productivity evolves according to a diffusion process

Rest of the Model

 $f(z,n) = z^{1-\alpha} n^{\alpha}$

 $dz = \mu(z)dt + \sigma(z)dW$



- Start from a discrete-time setup with time interval *dt*
- The firm's value function is

 $v(n,z) = \max\left\{v^*\right\}$

• the value of hiring is $v^*(n,z) = \max_{h \ge 0} \left(f(n,z) - wn - h \right)$

 $\pi(n,z)$

• the value of firing is

• the value of exit is v, as before

Start from Discrete Time

*
$$(n, z), \max\{\underline{v}, v^f(n, z)\}$$

$$\approx 1 - rdt$$

 $\cdot c_f - \Phi(h, n) dt + e^{-rdt} \mathbb{E} \left[v(n', z') \right]$

s.t. n' = n + hdt

$$v^{f}(n,z) = \max_{n^{f} \le n} v(n^{f},z)$$



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Continuous Time Limit

Add and subtract (1 - rdt)v(n, z) and defining $dv(n, z) \equiv v(n', z') - v(n, z)$, we have $v^*(n,z) = \max_{h \ge 0} \left(\pi(n,z) - \Phi(h,n) \right) dt + (1 - rdt) \mathbb{E} \left[dv(n,z) \right] + (1 - rdt) v(n,z)$ s.t. n' = n + hdt

• Apply Ito's lemma to dv(n, z): $dv(n,z) = v_n(n,z)dn + v_z(n,z)dn$ hdt Substitute (5) back into (4) and dropping dt² term $h \ge 0$ +v(n,z) - rdtv(n,z)

$$z)\left(\mu(z)dt + \sigma(z)dZ\right) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z)dt$$

 $v^*(n,z) = \max\left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z)\right)dt$





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Bellman Equation in Continuous Time

Therefore, we have

$$v(n,z) = \max\left\{v^*(n,z), \underline{v}^f(n,z)\right\}$$

$$v^{*}(n,z) = \max_{h \ge 0} \left(\pi(n,z) - \Phi(h,n) + v_{n}(n,z)h + v_{z}(n,z)\mu(z) + \frac{1}{2}\sigma(z)^{2}v_{zz}(n,z) \right) dt$$
$$+ v(n,z) - rdtv(n,z)$$
$$\underline{v}^{f}(n,z) = \max \left\{ \max_{n' \le n} v(n',z), \underline{v} \right\}$$

Three cases

- 1. Firms do not fire or exit: $v(n, z) > v^{j}$
- **2**. Firms fire workers: $v(n, z) = v^f(n, z)$
- 3. Firms exit: $v(n, z) = v > v^f(n, z)$, and

$$f(n, z) \text{ and } v(n, z) = v^*(n, z)$$

> v , and $v(n, z) > v^*(n, z)$
d $v(n, z) > v^*(n, z)$



Compactly, we can write

$$\min \left\{ rv(n,z) - \max_{h \ge 0} \left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z) \right), \right\} = 0$$

$$v(n,z) - \underline{v}^f(n,z)$$

- This is called HJB Quasi-Variational Inequality (HJB-QVI)

HJB-QVI

Distinct from HJB-VI because now the stopping value $v^f(n, z)$ is endogenous to v(n, z)





Policy Functions of HJB-QVI

$$\min\left\{ rv(n,z) - \max_{h \ge 0} \left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z) \right), \\ v(n,z) - \underline{v}^f(n,z) \right\} = 0$$

• When firms hire $(h \ge 0)$, the FOC impl

- Let h(n, z) denote the policy function
- The employment evolution is

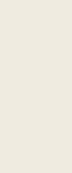
$$dn(n,z) = \begin{cases} h(n,z)dt & \text{if } n \leq n^{f}(n,z) \\ n^{f}(n,z) - n & \text{if } n > n^{f}(n,z) \end{cases}$$

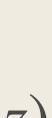
Let $\chi(n, z)$ denote an indicator function of exiting decision

lies
$$\partial_h \Phi(h, n) = v_n(n, z)$$

• When firms fire (h < 0), firms cut down employment to $n^{f}(n, z) = \arg \max_{n^{f} < n} v(n^{f}, z)$











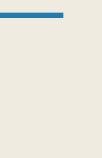


• When firms enter, they draw (n, z) from cdf $\Psi(n, z)$ We assume (potentially) inelastic entry:

$$m_t = M \times \left(\frac{1}{\bar{c}}\right)$$

Entry

 $\frac{1}{z^e}\int v(n,z)d\Psi(n,z)\right)^{\nu}$



(6)



Stationary Distribution

Define \mathscr{A}_{KFE} as the infinitesimal generator defined for a function f(n, z): $\mathscr{A}_{KFE}f(n,z) = \mu(z)f_z(n,z) + \frac{1}{2}$ $+\Lambda^f(n,z)[f(n$

where

$$\Lambda^{f}(n,z) = \begin{cases} \infty & \text{if } n \ge n^{f}(n,z) \\ 0 & \text{if } n < n^{f}(n,z) \end{cases}, \quad \Lambda^{e}(n,z) = \begin{cases} \infty & \text{if } \mathbb{I}^{e}(n,z) = 1 \\ 0 & \text{if } \mathbb{I}^{e}(n,z) = 0 \end{cases}$$

Let $\mathscr{A}_{KFF}^{\dagger}$ be adjoint operator of \mathscr{A}_{KFE} . The steady-state distribution g(n, z) satisfies

$$0 = \mathscr{A}^{\dagger}_{KFE} \mathscr{E}$$

$$-\sigma(z)^{2} f_{zz}(n,z) + (h(n,z) - sn) f_{n}(n,z)$$

$$h^{f}(n,z), z) - f(n,z) \Big] - \Lambda^{e}(n,z) f(n,z)$$

 $g(n,z) + m\psi(n,z)$

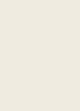


Equilibrium Definition

Equilibrium consists of $\{v(n, z), h(n, z), n^f(n, z), \chi(n, z), g(n, z), w, m\}$ such that

- 1. Value and policy functions { $v(n, z), h(n, z), n^f(n, z), \chi(n, z)$ } solve HJB-QVI
- 2. Stationary distribution g(n, z) solve KFE
- 3. Entry *m* is given by (6)
- **4**. Labor market clears: $\int \int ng(n, z) dn dz = L$

$(n, z), \chi(n, z), g(n, z), w, m$ such that $(n, z), n^{f}(n, z), \chi(n, z)$ solve HJB-QVI





Numerically Solving HJB-QVI – Nested Howard Algorithm







How to Solve HJB-QVI?

$$\min\left\{rv(n,z) - \max_{h\geq 0}\left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2v_{zz}(n,z)\right), v(n,z) - \underline{v}^f(n,z)\right\}$$

- - 1. optimization w.r.t. h
 - 2. optimization w.r.t. n^f inside $v^f(n, z)$
- Discrezie the state space n_1, \ldots, n_I and
- Use short-hand notation of, e.g., $v_{i,j} \equiv v(n_i, z_j)$

Relative to the case without adj. costs, there are two additional complications:

$$z_1, \ldots, z_J$$

We will use nested Howard's algorithm (Azimzadeh, Bayraktar, Labahn, 2018)





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No Firing or Exit

Start from the case where firms do not fire or exit, $\underline{v}_{i,i}^{f} = -\infty$

$$v(n,z) - \max_{h \ge 0} \left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z) \right) = 0$$

We can solve the above problem using Howard's algorithm:

- 1. Guess $v^k(n_i, z_j)$ for each (i, j)
- 2. Comp
- 3. Solve

bute optimal hiring using the FOC:

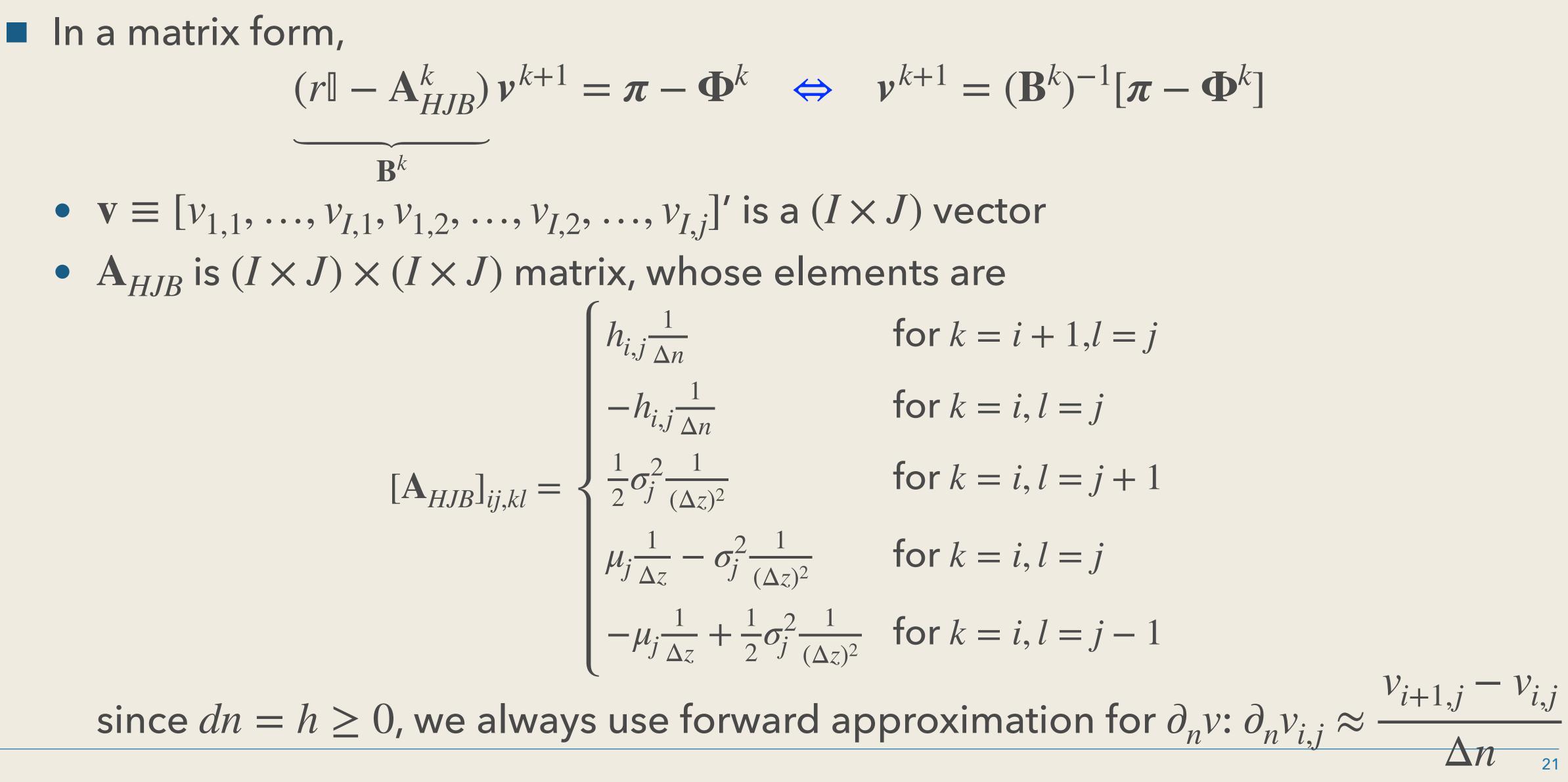
$$h_{i,j}^{k} = \max\{h^{*}, 0\} \text{ where } \partial_{h} \Phi(h^{*}, n_{i}) = \partial_{n} v_{i,j}^{k}$$
the linear system to obtain $v_{i,j}^{k+1}$

$$rv_{i,j}^{k+1} - \left(\pi_{i,j} - \Phi(h_{i,j}^{k}, n_{i}) + \partial_{n} v_{i,j}^{k+1} h_{i,j} + \mu_{j} \partial_{z} v_{i,j}^{k+1} + \frac{1}{2} \sigma_{j}^{2} \partial_{zz}^{2} v_{i,j}^{k+1}\right) = 0$$

4. Update $v_{i,j}^k := v_{i,j}^{k+1}$ and repeat until convergence



Linear System



$$\mathbf{P}^k \Leftrightarrow \mathbf{v}^{k+1} = (\mathbf{B}^k)^{-1}[\boldsymbol{\pi} - \boldsymbol{\Phi}^k]$$

' is a
$$(I \times J)$$
 vector
se elements are
for $k = i + 1, l = j$
for $k = i, l = j$
for $k = i, l = j + 1$
 $\sigma_j^2 \frac{1}{(\Delta z)^2}$ for $k = i, l = j - 1$
 $V = 1$



Howard Algorithm with Exit &
• For a fixed value of
$$\underline{v}^f \equiv [\underline{v}_{i,j}^f]_{i,j}$$
, we can incorporate exit & firing a
1. Guess v^0
2. For $k \ge 0$, given v^k , construct \mathbf{B}^k as described earlier, and set

$$d_{i,j} = \begin{cases} 0 \quad [\mathbf{B}^k v^k - \mathbf{\pi} - \mathbf{\Phi}]_i \le v_{i,j}^k - \underline{v}_{i,j}^f \\ 1 \quad [\mathbf{B}^k v^k - \mathbf{\pi} - \mathbf{\Phi}]_i > v_{i,j}^k - \underline{v}_{i,j}^f \end{cases}$$

3. Set

$$[\tilde{B}^{k}]_{ij,lm} = \begin{cases} [B^{k}]_{ij,lm} & \text{if } d_{i,j} = 0\\ [I]_{ij,lm} & \text{if } d_{i,j} = 1 \end{cases}, \qquad [q^{k}]_{i,j} = \begin{cases} [\pi - \Phi]_{i,j} & \text{if } d_{i,j} = 0\\ \underbrace{\nu^{f}_{i,j}} & \text{if } d_{i,j} = 1 \end{cases}$$

4. Update v^{k+1} solving

 $\tilde{\boldsymbol{B}}^k \boldsymbol{v}^{k+1} = \boldsymbol{q}^k$

Firing

as follows

$$\Leftrightarrow v^{k+1} = [\tilde{B}^k]^{-1} q^k$$



Nested Howard's Algorithm

The outer loop keeps updates v^f starting from $v^f = -\infty$

- 1. Set the value of firing to $v_{i,i}^{f,0} = -\infty$ for all i, j
- **2.** For each k = 0, 1, ...
 - i. Given $\{v_{i,i}^{f,k}\}_{i,j'}$ set $\underline{v}_{i,j}^f \equiv \max\left\{\underline{v}, v_{i,j}^{f,k}\right\}$
 - ii. Given $\underline{v}^f \equiv [\underline{v}^f_{i,i}]_{i,j}$, solve HJB-VI (not QVI) using Howard's algorithm
 - iii. Compute the new value of firing as
 - If $v_{i,j}^{f,k+1}$ is close enough to $v_{i,j}^{f,k}$, we are done.
 - Otherwise, set $v_{i,j}^{f,k} := v_{i,j}^{f,k+1}$ and go back to 2.i.

- $v_{i,j}^{f,k+1} = \max_{i' < i} v_{i,j}$



- Some use algorithms that simultaneously update v^{f} in inner loop
- Never do this. I wasted my entire summer because of it.
- At the same time, the nested Howard algorithm is inefficient
 - Need many outer loop iterations to converge
- Alternative algorithms that improve speeds have been proposed:
 - The most successful one seems to be penalized Howard algorithm (Azimzadeh and Forsyth, 2016; Azimzadeh, Bayraktar, and Labahn, 2018)
 - I tried to implement it but failed
- If you implement Penalized Howard's algorithm, I will count it as a final project

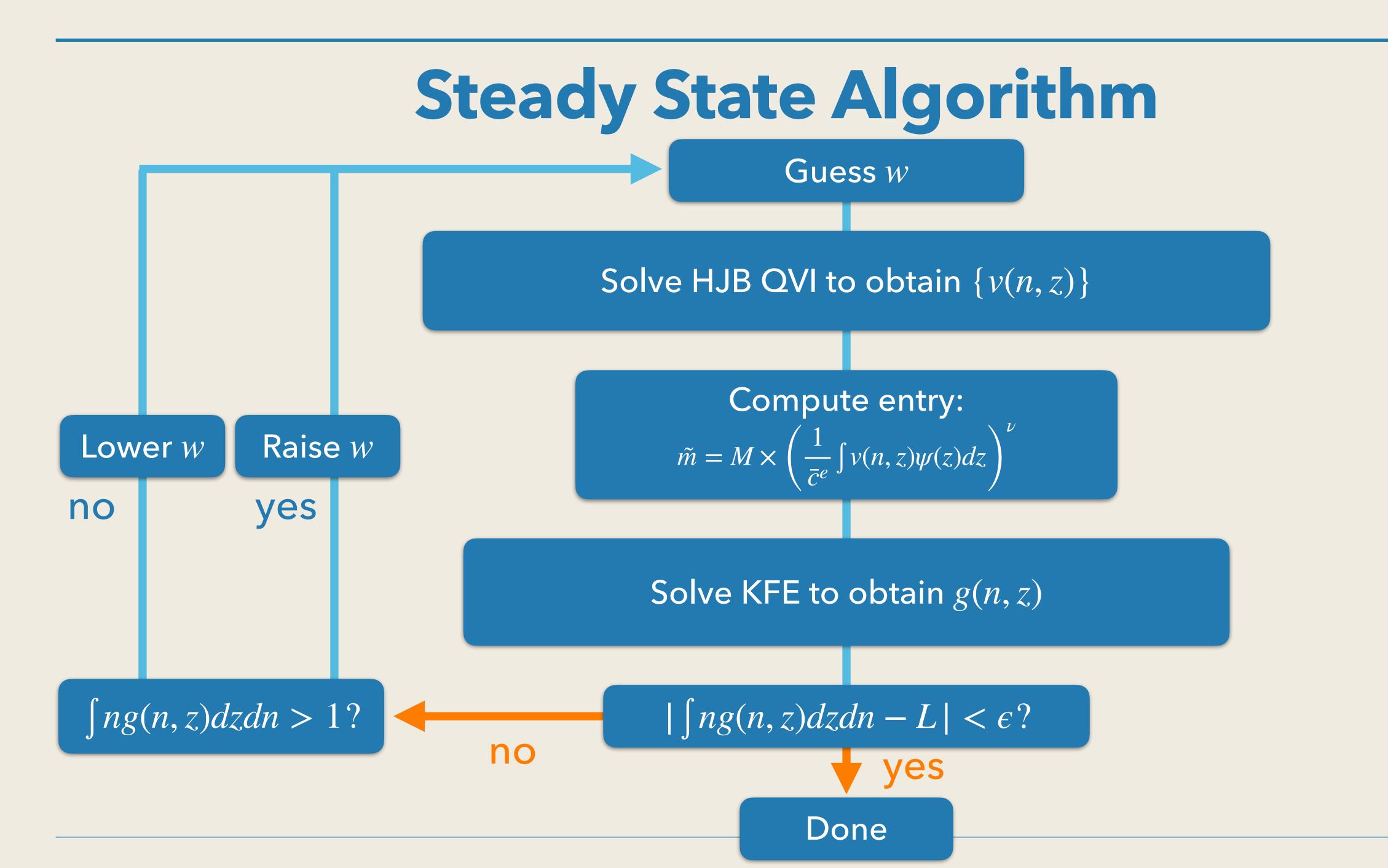
Can We Do Better?



Numerically Computing Steady State Equilibrium









Discretized Kolmogorov Forward Eq

$$\begin{bmatrix} \boldsymbol{D} + (\boldsymbol{A}_{HJB}\boldsymbol{M})' \end{bmatrix} \boldsymbol{g} + m\boldsymbol{\psi} = 0$$

$$\begin{bmatrix} \boldsymbol{M} \end{bmatrix}_{ij,kl} = \begin{cases} 1 & \text{for } n^f(n_i, z_j) = n_k, \chi(n_i, z_j) = 0, l = j \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} \boldsymbol{D} \end{bmatrix}_{ij,kl} = \begin{cases} 1 & \text{for } i = k, j = l, n^f(n_i, z_j) < n_i, \chi(n_i, z_j) = 1 \\ 0 & \text{otherwise} \end{cases}$$

• The matrix **D** ensures that $[g]_{ij} = 0$ for states (n_i, z_j) that are never reached

quation

• The matrix *M* takes care of transitions associated with jump in the state variables





Macroeconomic Implications of Slow to Hire, Quick to Fire







 $\Phi(h,n)$

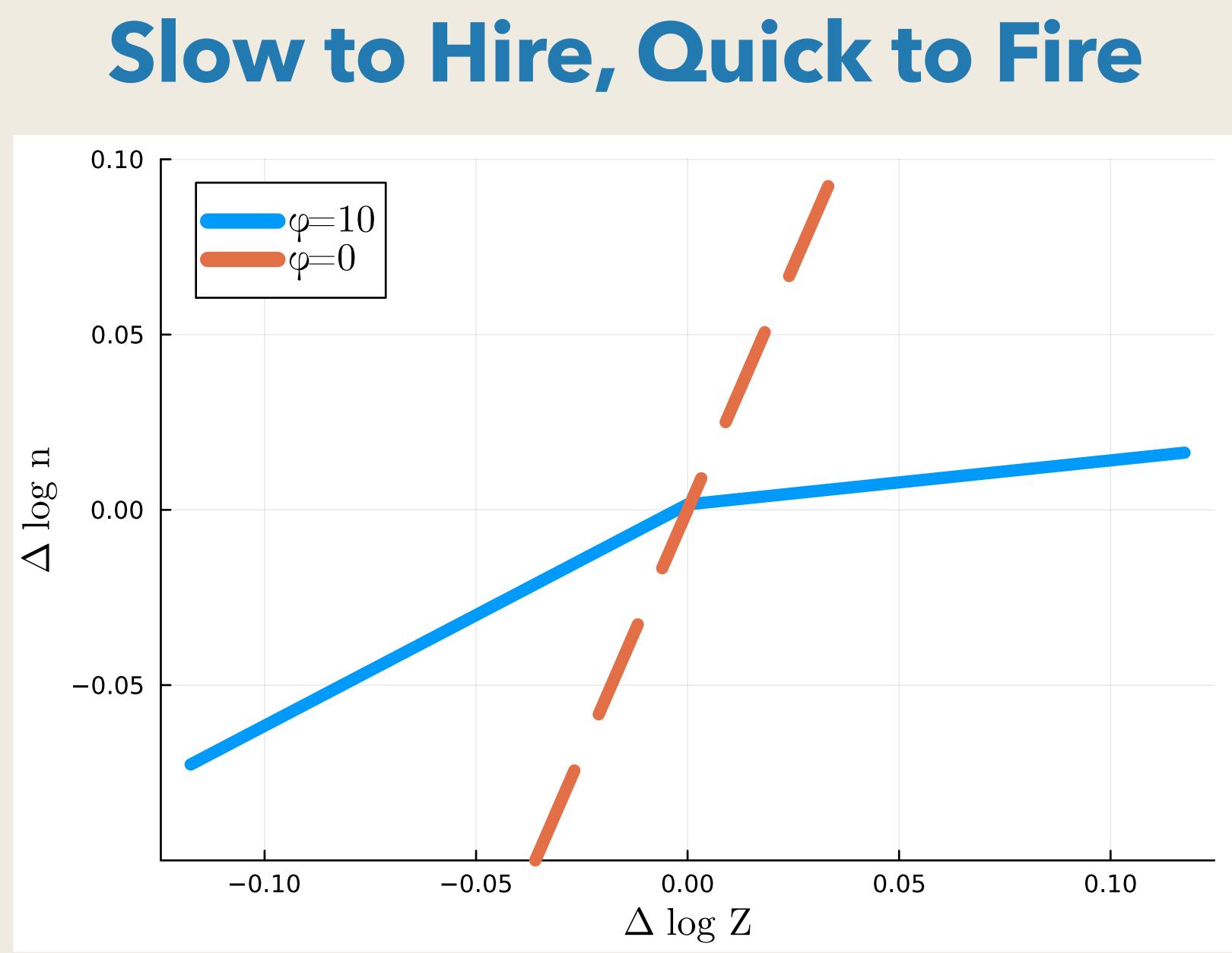
I set $\phi = 10$ and contrast with $\phi = 0$

All the other parameters are unchanged from the lecture note 2

Parameterization

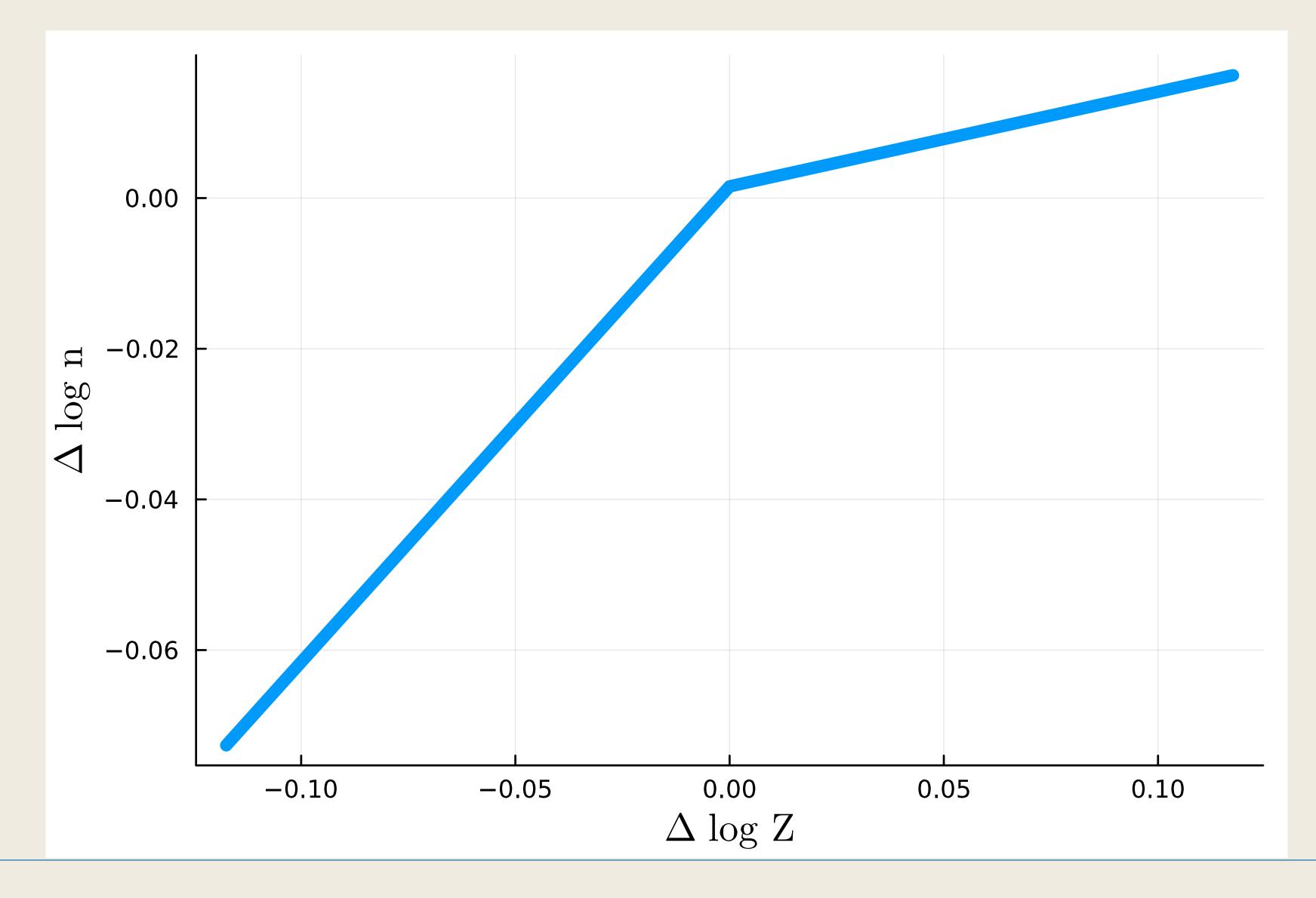
$$=\frac{\phi}{2}\left(\frac{h}{n}\right)^2 n$$





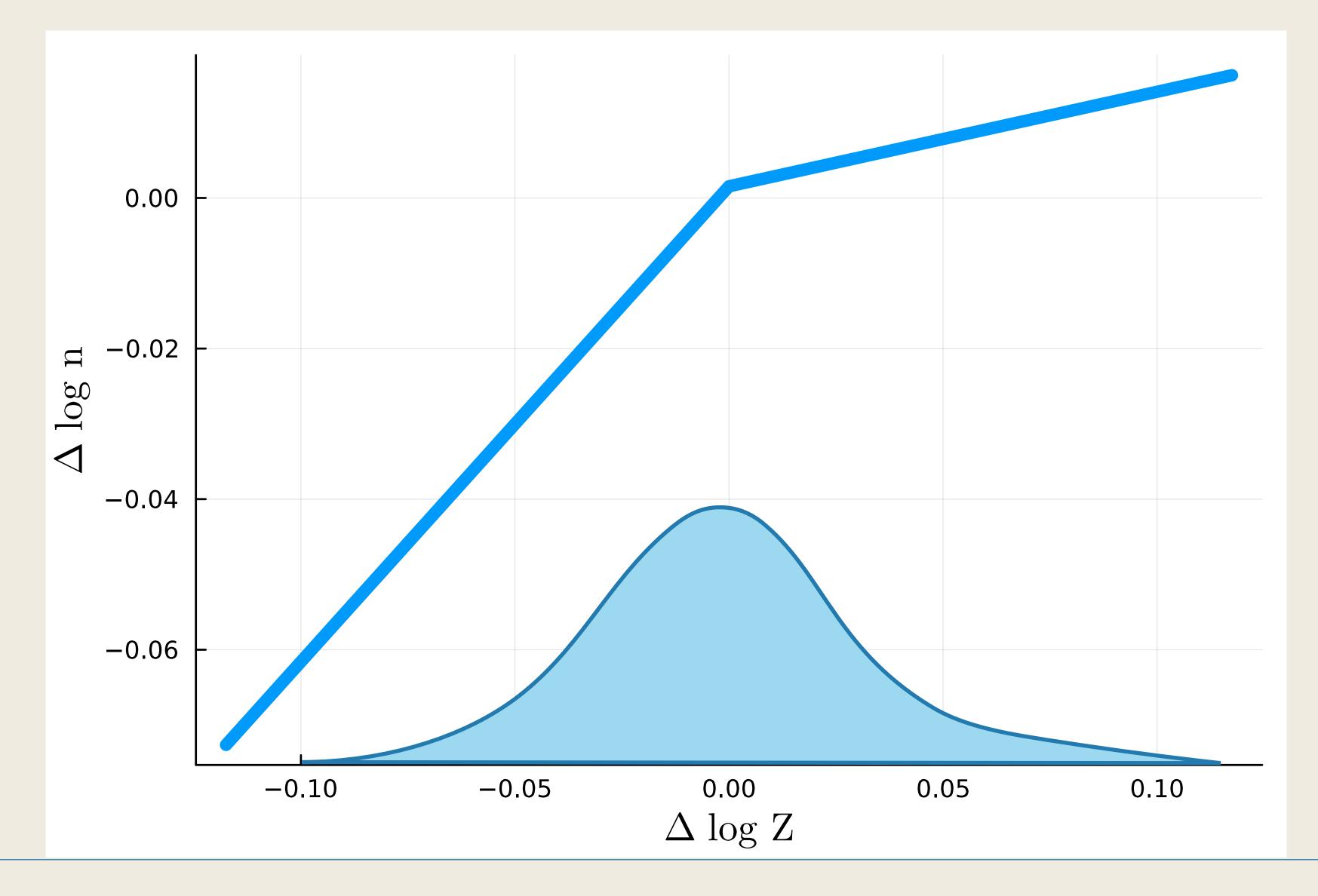


SS Distribution of Employment Growth



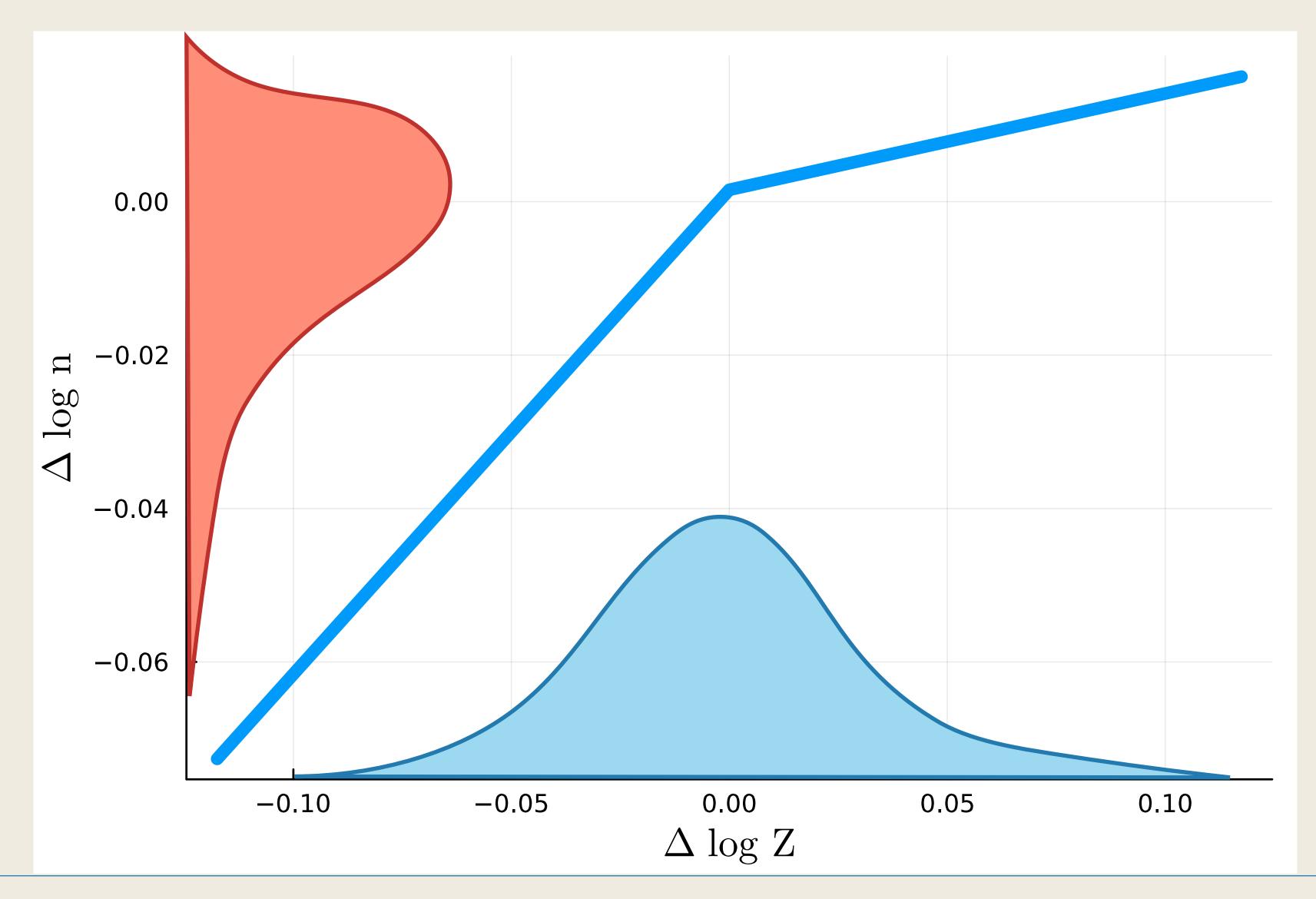


SS Distribution of Employment Growth



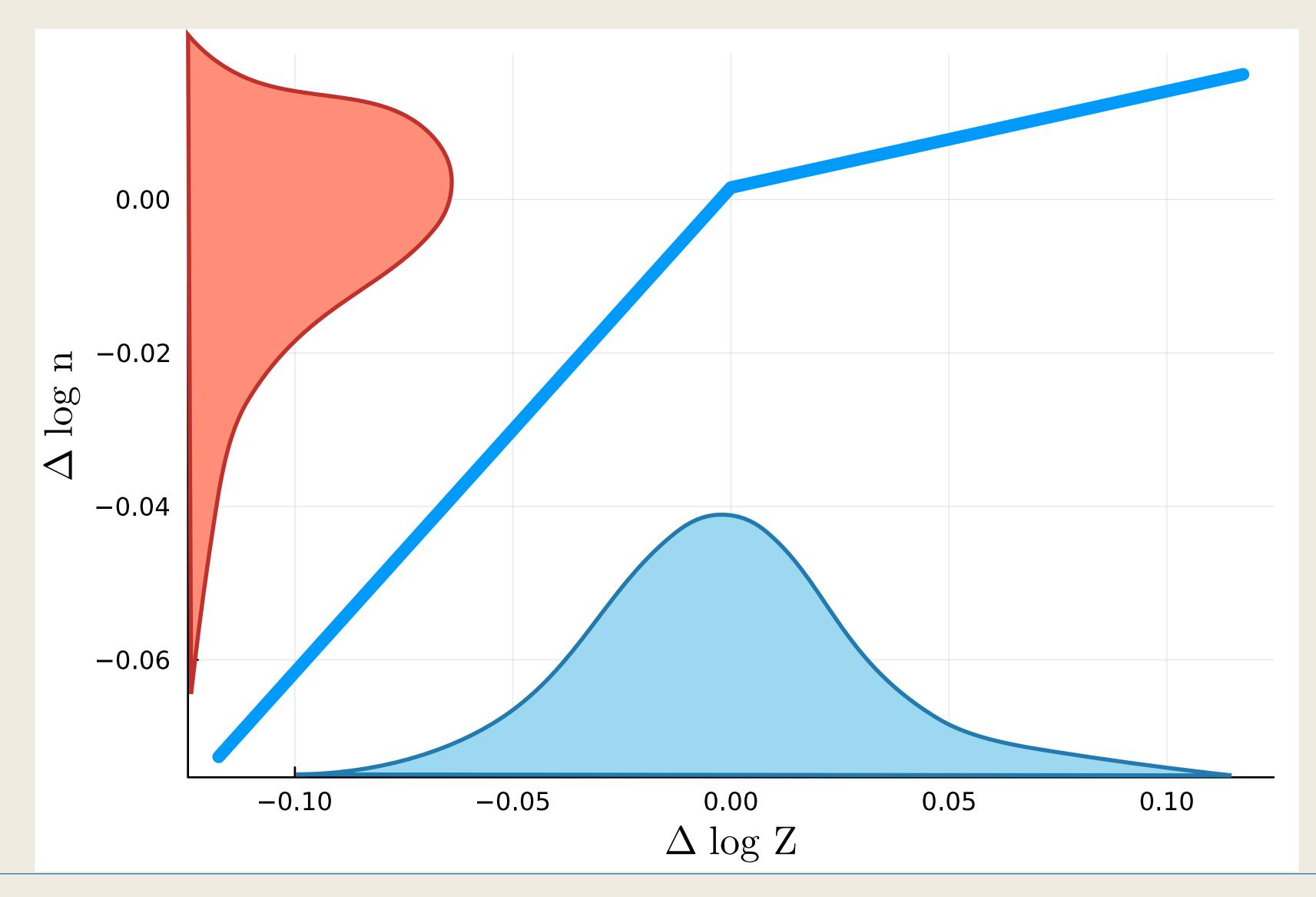


SS Distribution of Employment Growth



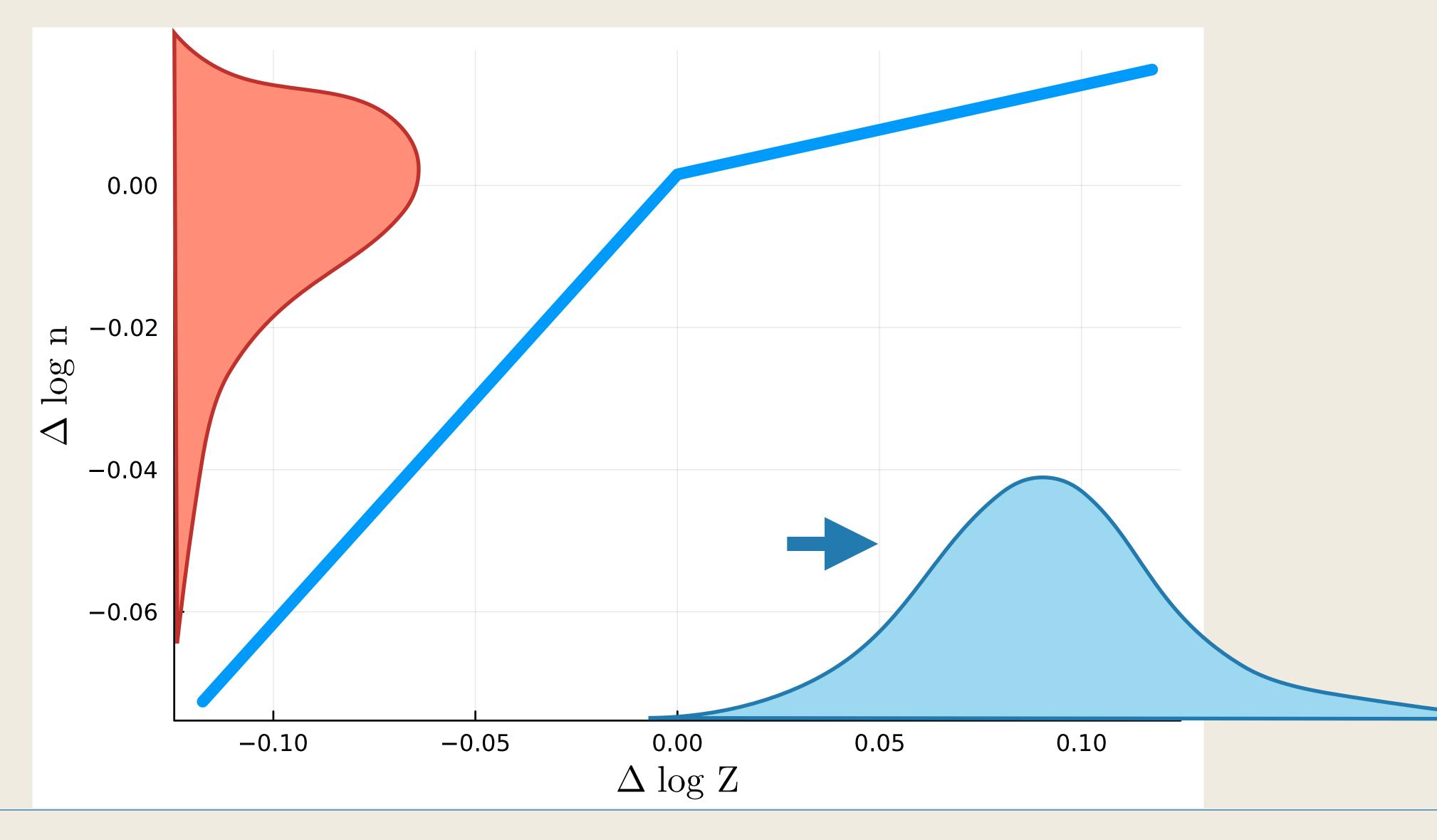


Aggregate Positive Productivity Shocks



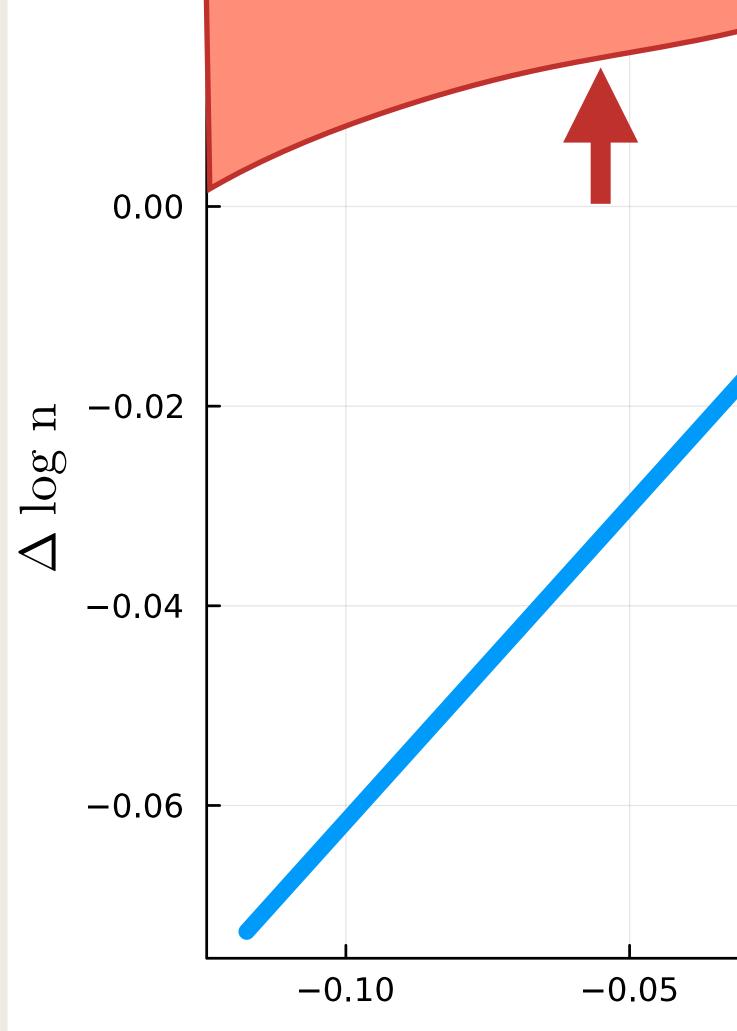


Aggregate Positive Productivity Shocks



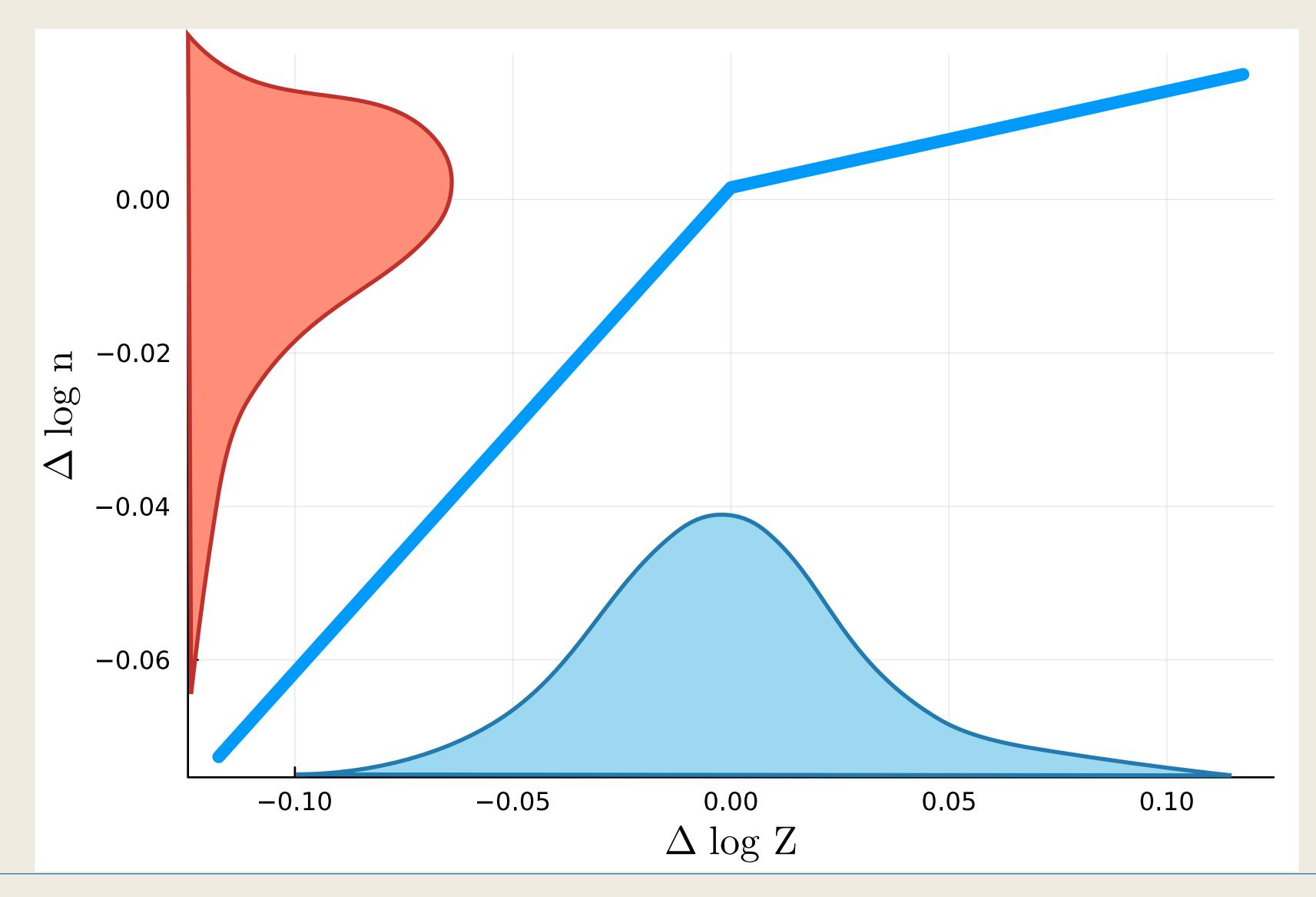


Aggregate Positive Productivity Shocks 0.00 -0.02 \mathbf{n} $\Delta \log$ -0.04-0.06 -0.05 0.00 0.05 -0.100.10 $\Delta \log Z$



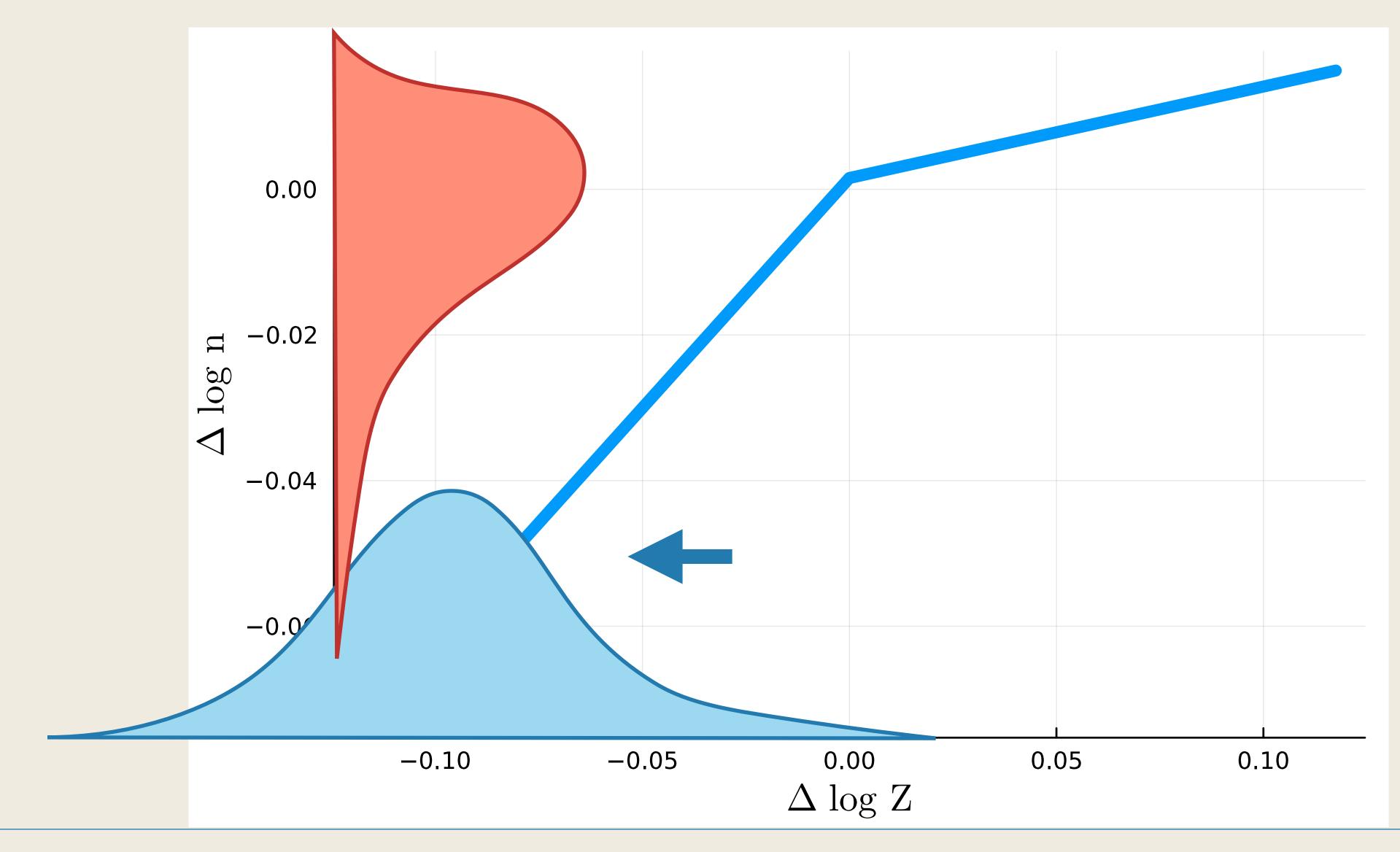


Aggregate Positive Productivity Shocks



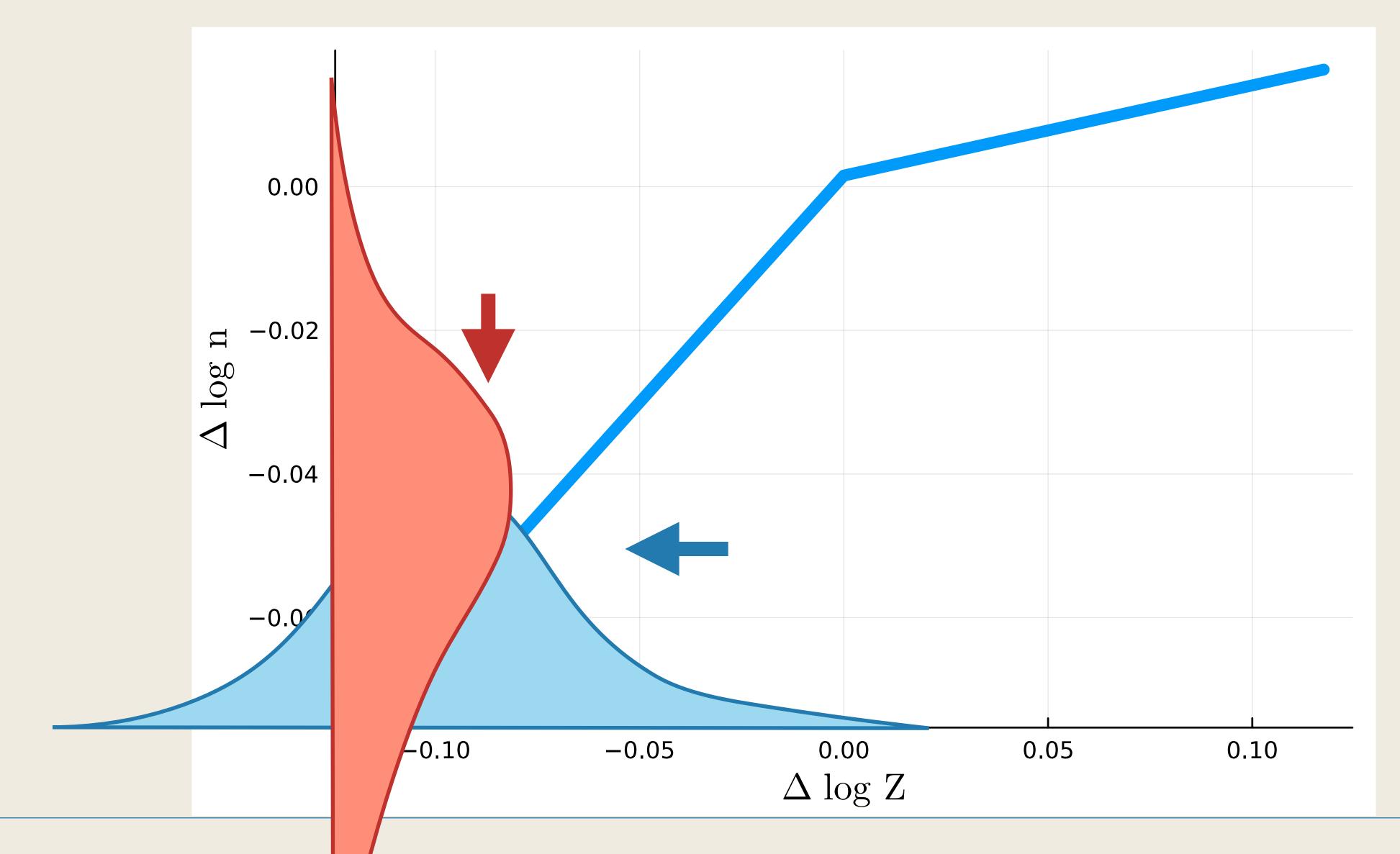


Aggregate Positive Productivity Shocks

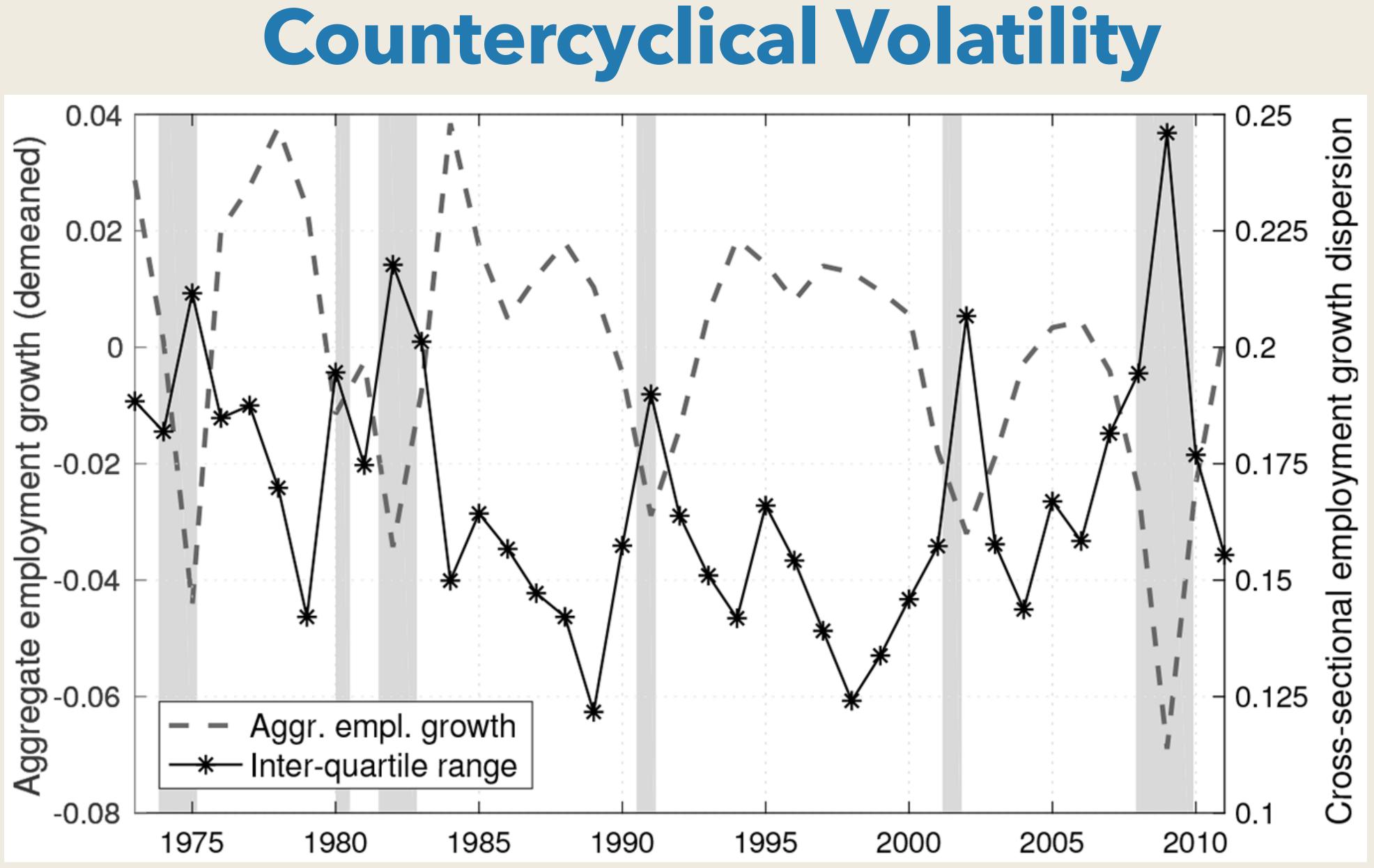




Aggregate Positive Productivity Shocks







Source: Ilut, Kehrig & Schneider (2018)



Countercyclical Volatility & Skewness

Moments

Data Linear hiring Concave hiring

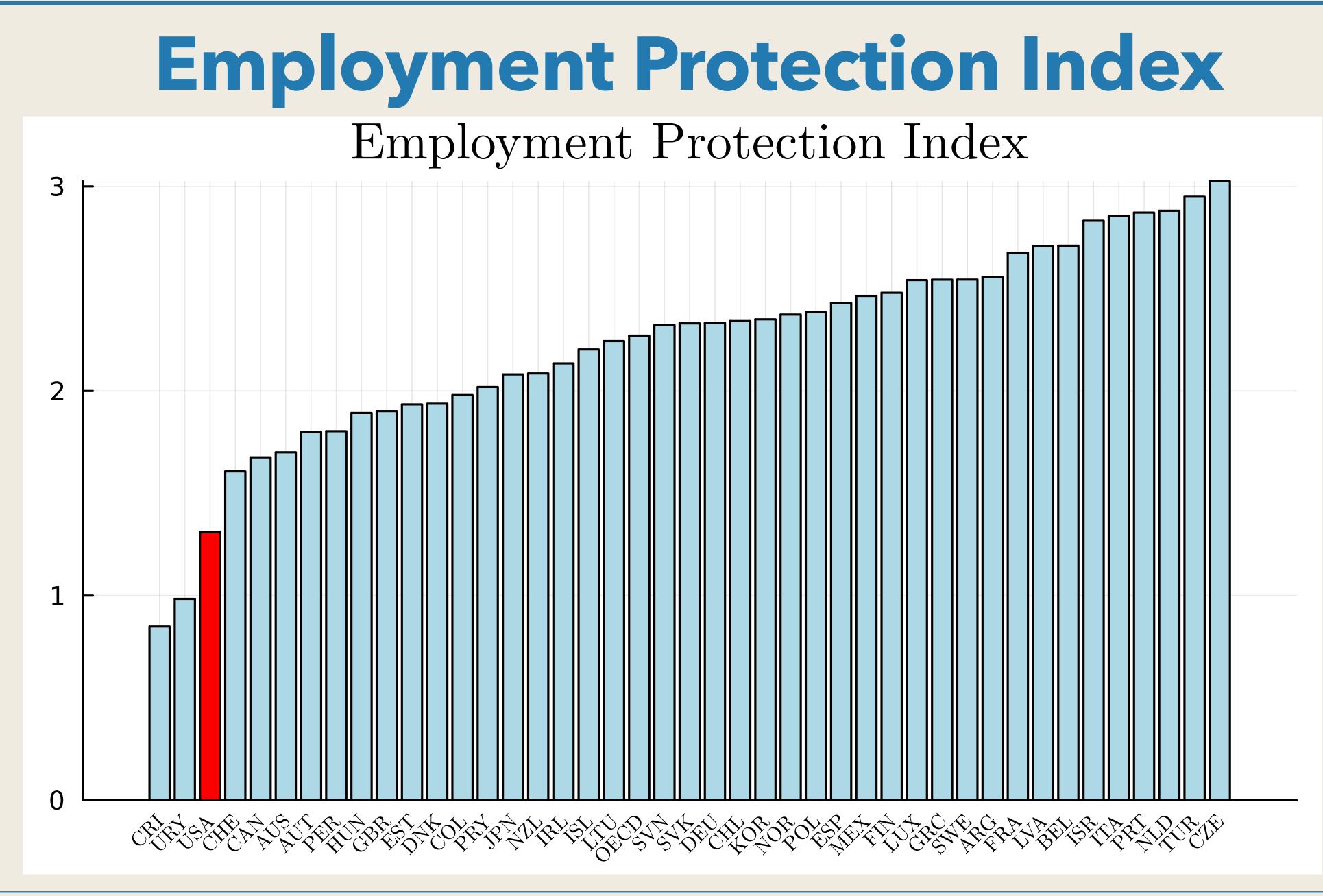
Source: Ilut, Kehrig & Schneider (2018)

$$\frac{IQR(n^{i}|u^{a} = -\sigma^{a})}{IQR(n^{i}|u^{a} = +\sigma^{a})}$$
(1)



Firing Cost and Misallocation – Hopenhayn & Rogerson (1993)





Data source: OECD





- What is the cost of strict firing regulations?
- - US: $\tau = 0$
 - Europe: high τ
- Firing costs take the form of taxes

Suppose that in order to fire a worker, firms have to pay $\tau \times$ annual wage salary

The collected tax revenue is rebated back to households as lump-sum transfers

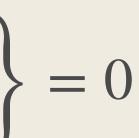


$$\min\left\{rv(n,z) - \max_{h\geq 0} \left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2 v_{zz}(n,z)\right), v(n,z) - \underline{v}^f(n,z)\right\}$$

$$\underline{v}^{f}(n,z) = \max\left\{\max_{n^{f} \le n} v(n^{f},z) - \tau w(n-n^{f}), \underline{v}\right\}$$

- This is the only modification
- No firing tax when exiting (maybe I should have assumed otherwise)

HJB-QVI



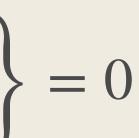


$$\min\left\{rv(n,z) - \max_{h\geq 0}\left(\pi(n,z) - \Phi(h,n) + v_n(n,z)h + v_z(n,z)\mu(z) + \frac{1}{2}\sigma(z)^2v_{zz}(n,z)\right), v(n,z) - \underline{v}^f(n,z)\right\}$$

$$\underline{v}^{f}(n, z) = \max \left\{ \max_{n^{f} \le n} v(n^{f}, z) + \tau w(n - n^{f}), \underline{v} \right\}$$
firing tax

- This is the only modification
- No firing tax when exiting (maybe I should have assumed otherwise)

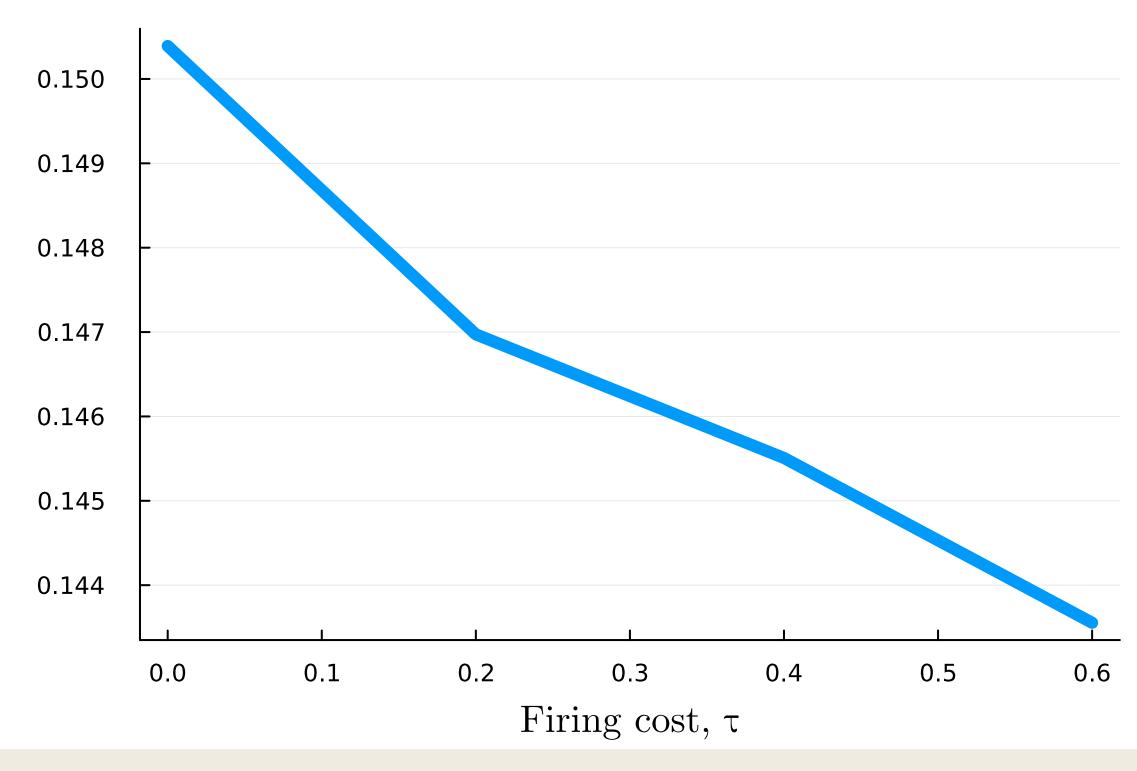
HJB-QVI





Misallocation Cost of Firing Regulations

Wage

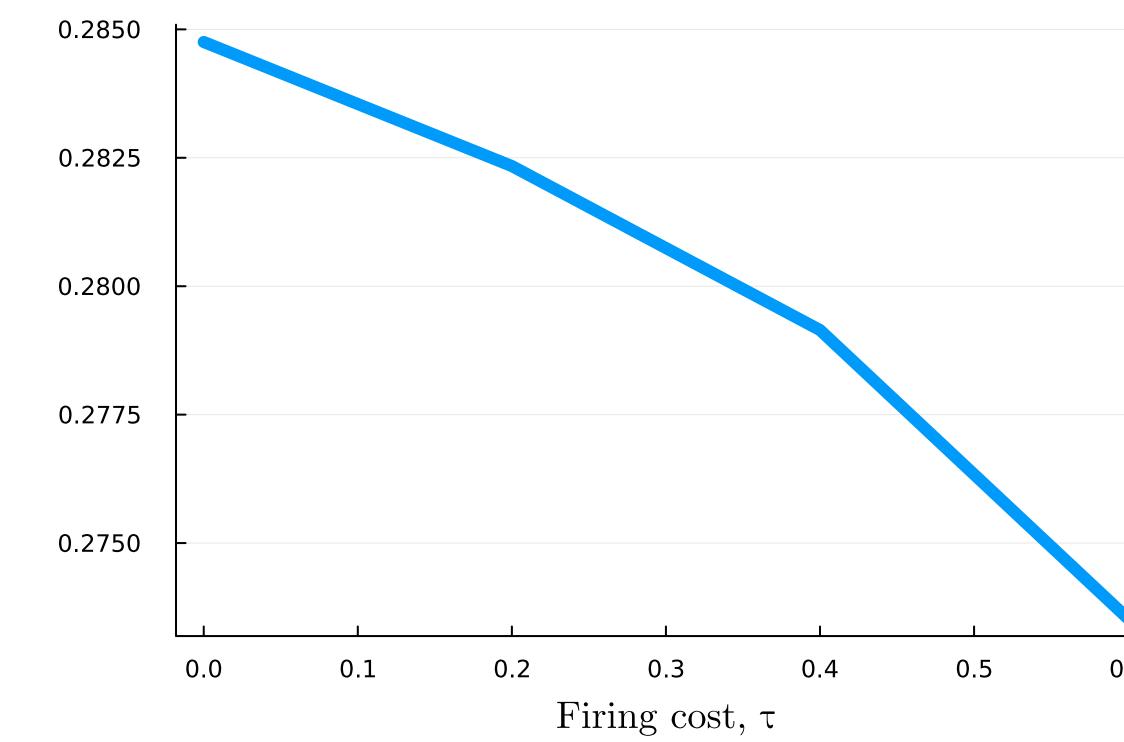


Firing costs lead to the misallocation of workers because

1. Unproductive firms cannot downsize

2. Productive firms become hesitant to expand

Labor Productivity







Idiosyncratic Distortion

- Firing tax is a distortion at the aggregate level
- There is no shortage of reasons to expect firms to face idiosyncratic distortions
 - Corruption, firm-level taxes/subsidies, financial frictions, incomplete contracts
- Restuccia & Rogerson (2008) consider wedges in the form of

(1 +

where
$$\tau_i = \begin{cases} \tau & \text{with prob } 1/2 \\ -\tau & \text{with prob } 1/2 \end{cases}$$

 τ_i is assigned when firm *i* is born and fixed over time

$$(\tau_i)z^{1-\alpha}n^{\alpha}$$



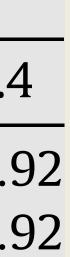
Misallocation from Idiosyncratic Distorton

Table 3

Effects of idiosyncratic distortions—uncorrelated case

Variable	$ au_t$			
	0.1	0.2	0.3	0.4
Relative Y	0.98	0.96	0.93	0.9
Relative TFP	0.98	0.96	0.93	0.9

Source: Restuccia & Rogerson (2008)





Non-Parametric Identification of Misallocation

– Carrillo, Donaldson, Pomeranz, & Singhal (2023)



What is the Cost of Misallocation?

- How large is the cost of misallocation in the data?
- Let us step back and consider a static model with a fixed mass of firms
- Each firm *i* produces using
- The efficient allocation solves
- The solution features equalization of MPL:

- $y_i = f_i(n_i)$
- $Y^* \equiv \max_{\{n_i\}} \int f_i(n_i) di$
 - s.t. $\int n_i di = L$
- $f'_i(n_i) = w$ for all *i*





Variance of MPL

- $\frac{Y Y^*}{Y^*} \approx -\frac{1}{2}$
 - where $MPL_i = f'_i(n_i)$, $\lambda_i = w_i n_i / Y^*$ and $\epsilon_i \equiv -\frac{d \log MPL_i}{d \log n_i}$
- (Weighted) variance of MPL is the key moment for the cost of misallocation
- Testing the presence of misallocation \Leftrightarrow testing Var(MPL_i) = 0
- How do we get the distribution of MPL?
 - 1. Assume $f_i(n_i) = Z_i n_i^{\alpha}$, and then $MPL_i = \alpha \frac{y_i}{n_i}$ (Hsieh & Klenow, 2009)
 - 2. Nonparametrically identify the distribution of MPL (Carrillo et al. 2023)

Take arbitrary allocation $\{n_i\}$. Up to a second order around the efficient allocation

$$\frac{1}{2}\int \lambda_i \epsilon_i \log(MPL_i/w)^2 di$$



Nonparametric Identification

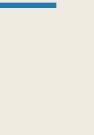
- Taking the first-order approximation of equation (7), $\Delta y_i = \beta_i \Delta n_i + \epsilon_i$
 - ϵ_i : technology shocks (i.e., changes in $f_i(\cdot)$)
 - $\beta_i = f'_i(n_i) = MPL_i$: treatment effect of exogenously increasing n_i on y_i
- With suitable instruments Z_i that exogenously shift n_i , $\mathbb{E}[\beta_i^k]$ (k = 1, 2, ...) are identified (Masten & Torgovitsky, 2016)





Empirical Implementation

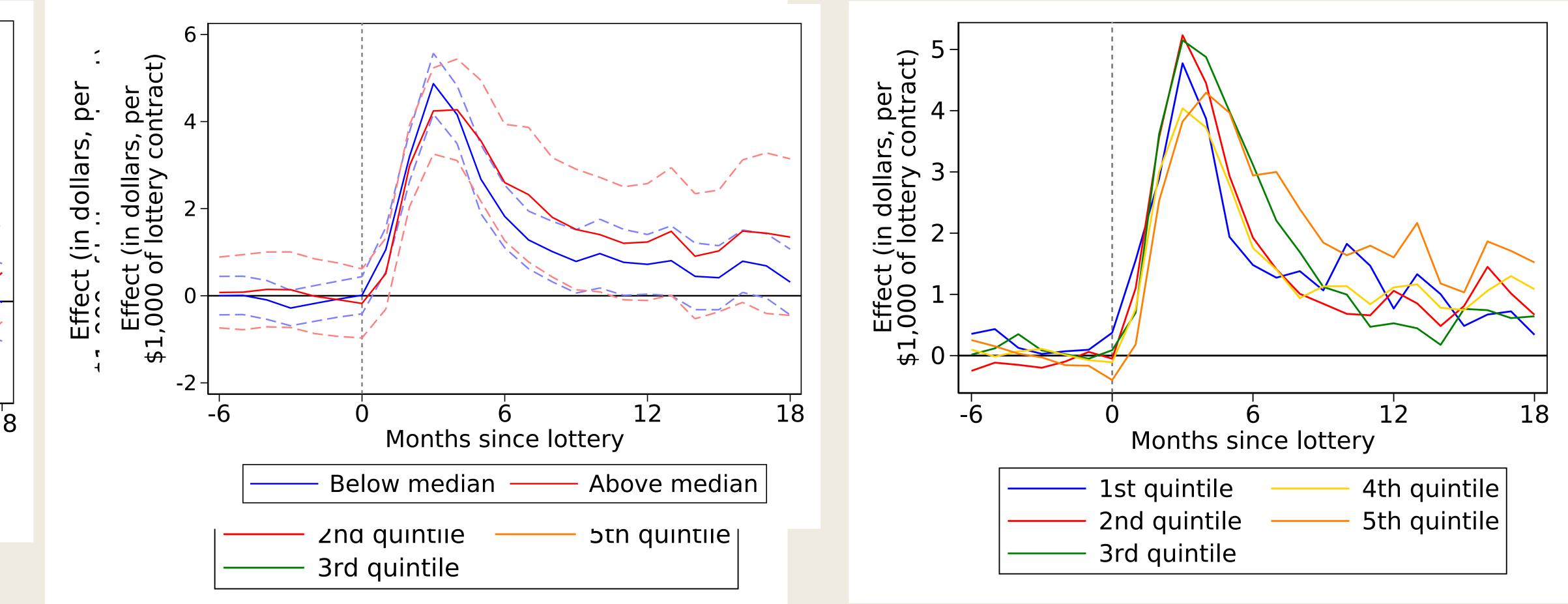
- Construction sector in Ecuador, 2009-2014
- Public construction projects were allocated through a randomized lottery
- Lottery serves as an ideal instrument
 - exogeneity: orthogonal to technology shocks ϵ_i or MPL_i
 - relevance: winning a lottery does shift n_i



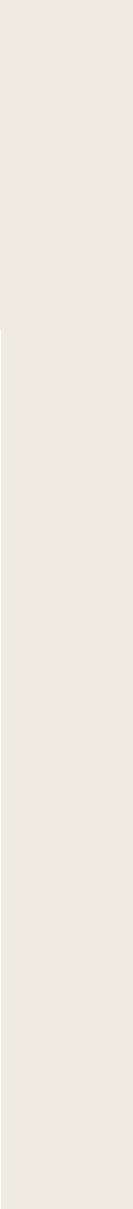
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Heterogenous Treatment Effects by Firm Size?

Sales



Labor Inputs





Small Cost of Misallocation

Table 4: Estimated Cost of Misallocation

Panel (a): IVCRC estimates Baseline

Panel (b): Alternative procedure assur Constant returns-to-scale ($\gamma = 1$)

• Assume $\epsilon_i = 3$ for all i

- The welfare cost of misallocation is 1.6%

$\mathbb{E}_{ar{\lambda}}[ar{\mu}] \ (1)$	$\mathbb{V}ar_{\overline{\lambda}}[\overline{\mu}]$ (2)	$\frac{\Delta W}{W}$ (3)			
	(-)	(0)			
1.126	0.014	0.016			
1.093, 1.161]	[0, 0.341]	[0, 0.261]			
ming common scale elasticities					
1.240	0.611	0.479			
1.223, 1.257]	[0.544, 0.730]	[0.427, 0.572]			

[]

Hsieh-Klenow type calculation implies 48% of welfare loss in the same dataset





- Laissez-faire of Hopenhayn-Rogersion with labor adjustment costs is efficient
- But, MPL is not equalized in a static sense
- Firms hire workers until (present discounted value of hiring a worker) = (hiring cost today)
- Hiring a worker is an investment
- How do we incorporate dynamics without imposing strong assumptions? How do we incorporate entry & exit dynamics?

Questions

