# Sorting and Wages in the Labor Market

## 741 Macroeconomics
## Topic 4

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# Sorting in the Labor Market

- In all the models we considered so far, there is no sorting
- Good workers are equally likely to work for good firms and bad firms
- In the data, “good” workers are substantially more likely to work for “good” firms
  - Recall findings from AKM and BLM in lecture note 1
- Why?

# Frictionless Model of Matching

– Becker (1974)

# Environment

- Continuum of workers divided into type $z$ with mass $m_z$
- Continuum of firms divided into type $p$ with mass $m_p$
- A firm of type $p$ hiring a type $z$ worker produces $f(z, p)$
- Competitive labor market with wage $w(z)$ for type $z$ workers
- A firm chooses which worker to hire, taking $w(z)$ as given:

$$\zeta(p) = \arg\max_{z} f(z, p) - w(z)$$

- What can we say about the matching pattern $\zeta(p)$?

# Positive Assortative Matching

Suppose $f$ is supermodular:

$$\partial_{z,p} f(z, p) \geq 0.$$

Then, $\zeta(p)$ is increasing in $p$. That is, there is positive assortative matching (PAM).

- Proof immediately follows from Topkis' monotonicity theorem
- If high $z$ and high $p$ are complementary, high $z$ is matched with high $p$
- Negative assortative matching (NAM) obtains under $\partial_{z,p} f(z, p) \leq 0$

# Matching with Search Friction

– Shimer & Smith (2000)

# Why Search Friction?

Becker model predicts that…

1. There cannot be unmatched agents on both sides
2. No mismatch $\Rightarrow$ all the job-to-job transitions come from a change in fundamentals
3. No wage dispersion within a firm

Search theory gives a natural resolution for all of them

# Environment

- Continuous time, $t \in [0,\infty)$
- Workers with discrete types $z$ and associated mass of $m_z$
- Jobs (firms) with discrete types $p$ and associated mass of $n_p$
  - Differently from earlier lecture notes, assume vacancies are durable
  - If workers quit/separate, the job becomes vacant
- All agents are risk-neutral with discount rate $\rho$
- For notational simplicity, assume flow value of unemployment & vacancy are zero
- A match $(z, p)$ produces $f(z, p)$ units of output

# Search Friction

- Search is random, and no on-the-job search for now
- The matching function is

$$M\left(\sum_z u_z, \sum_p v_p\right)$$

- The rate at which an unemployed worker meets with type $p$ firm is

$$\underbrace{\frac{M\left(\sum_z u_z, \sum_p v_p\right)}{\sum_z u_z} \times \frac{1}{\sum_p v_p}}_{\chi_0} v_p \equiv \chi_0 v_p$$

- Likewise, the meeting rate of a vacancy with type $z$ worker is $\chi_0 u_z$
- We will treat $\chi_0$ as parameters for most part (can always find $M$ to hit any $\chi_0$)
- All matches exogenously separate at rate $\delta$

# Value Functions

- Value functions:

$$\rho U_z = \chi_0 \sum_p v_p \max\{W_{z,p} - U_z, 0\}$$

$$\rho W_{z,p} = w_{z,p} + \delta(U_z - W_{z,p})$$

$$\rho V_p = \chi_0 \sum_z u_z \max\{J_{z,p} - V_p, 0\}$$

$$\rho J_{z,p} = f(z,p) - w_{z,p} + \delta(V_p - J_{z,p})$$

- Joint match surplus, $S_{z,p} \equiv W_{z,p} + J_{z,p} - U_z - V_p$, follows

$$(\rho + \delta) S_{z,p} = f(z,p) - \rho U_z - \rho V_p$$

# Nash Bargaining

- Assume wage is determined by Nash bargaining with worker bargaining power $\gamma$:

$$\max_{w_{z,p}} (W_{z,p} - U_z)^{\gamma}(J_{z,p} - V_p)^{1-\gamma}$$

- This results in

$$W_{z,p} = U_z + \gamma S_{z,p}$$

$$J_{z,p} = U_z + (1-\gamma) S_{z,p}$$

# Equilibrium Conditions

- $\{S_{z,p}, V_p, U_z\}$ solve

$$\rho U_z = \chi_0 \gamma \sum_p v_p \max\{S_{z,p}, 0\}$$

$$\rho V_p = \chi_0 (1-\gamma) \sum_z u_z \max\{S_{z,p}, 0\}$$

$$(\rho + \delta) S_{z,p} = f(z,p) - \rho U_z - \rho V_p$$

- The steady state distribution $\{\phi_{z,p}, u_z, v_p\}$ satisfy

$$\delta \phi_{z,p} = \chi_0 u_z v_p \mathbb{I}[S_{z,p} > 0]$$

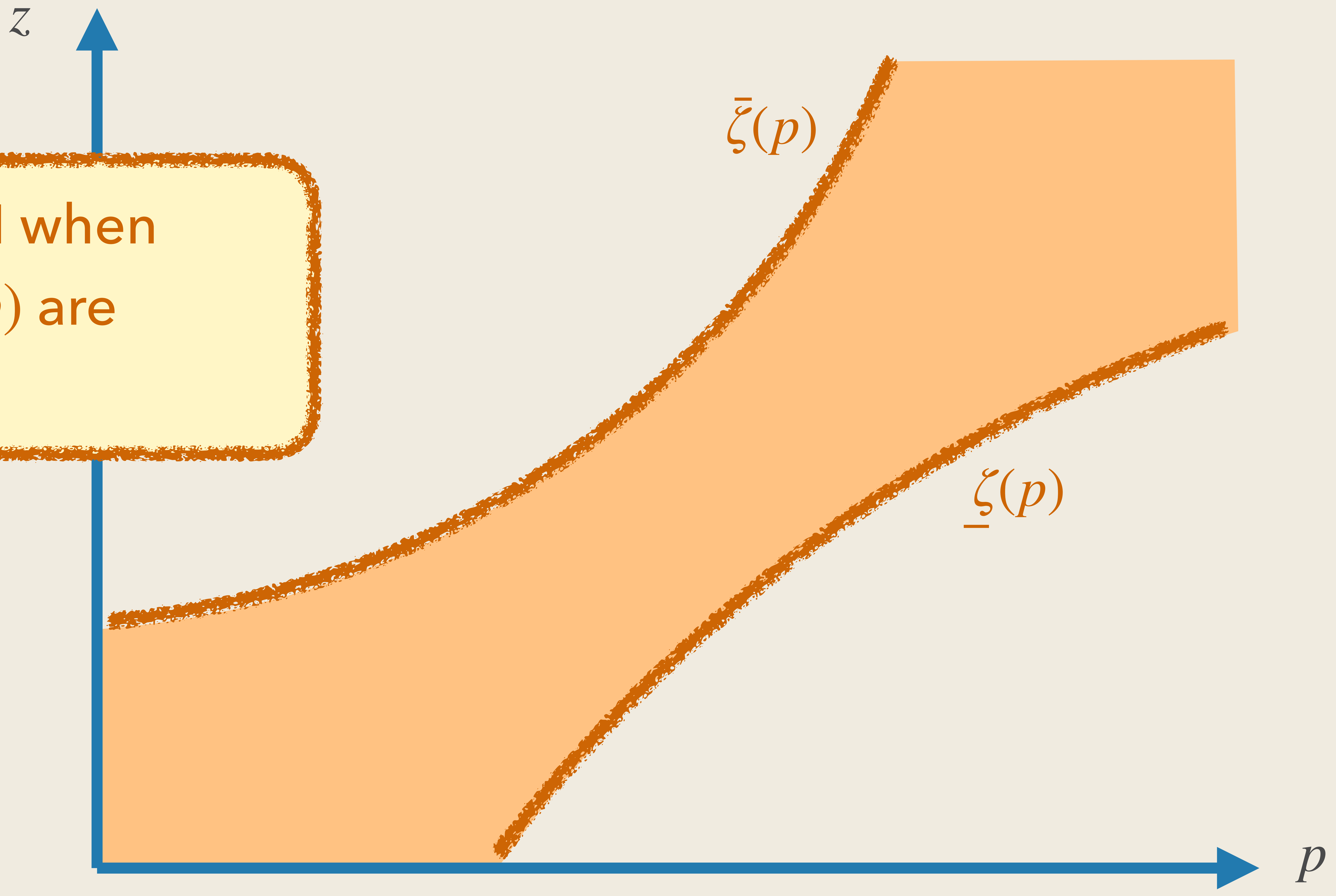
$$u_z = m_z - \sum_p \phi_{z,p}, \quad v_p = n_p - \sum_z \phi_{z,p}$$

- Wage is given by $w_{z,p} = \rho U_z + \gamma(\rho + \delta) S_{z,p}$

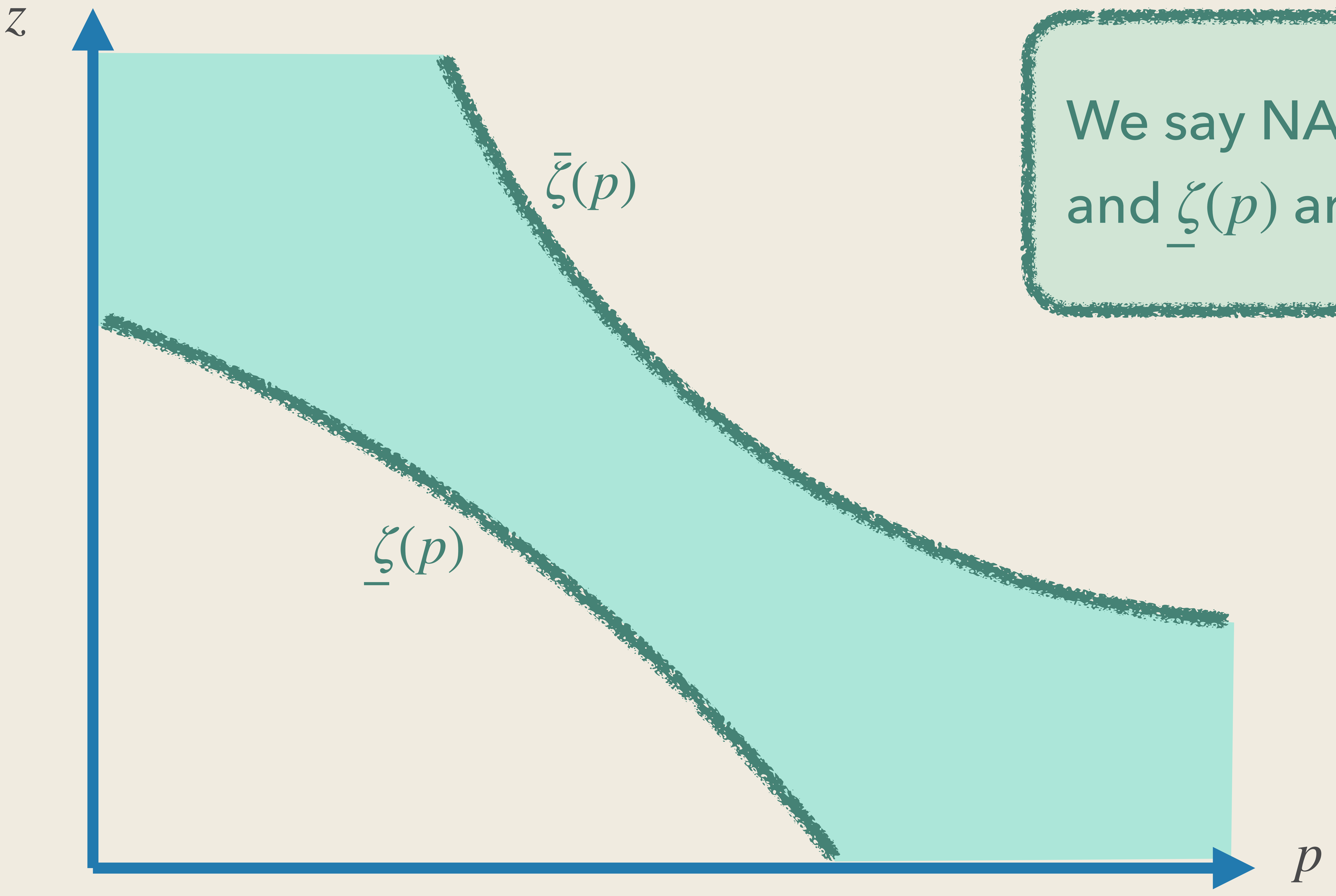
# Frictionless Matching

# PAM

We say PAM when $\bar{\zeta}(p)$ and $\underline{\zeta}(p)$ are increasing



# NAM




We say NAM when $\bar{\zeta}(p)$ and $\underline{\zeta}(p)$ are decreasing

# Equilibrium Properties

- Does equilibrium exist? – Yes! (Shimer & Smith, 2000)
- Is equilibrium unique? – Not in general (Shimer & Smith, 2000)
- Matching pattern (Shimer & Smith, 2000):
  - $\partial_{zp} f \geq 0, \partial_{zp}[\log f_z] \geq 0, \partial_{zp}[\log f_{zp}] \geq 0 \Rightarrow$ PAM
  - $\partial_{zp} f \leq 0, \partial_{zp}[\log f_z] \leq 0, \partial_{zp}[\log f_{zp}] \leq 0 \Rightarrow$ NAM
- With search friction, a stronger condition is needed than the frictionless case
  - Even when $\partial_{zp} f > 0$, "good" and "middle" form a match due to search friction
  - "Middle" and "middle" then may decide not to form a match
  - But "Low" and "Middle" instead form a match

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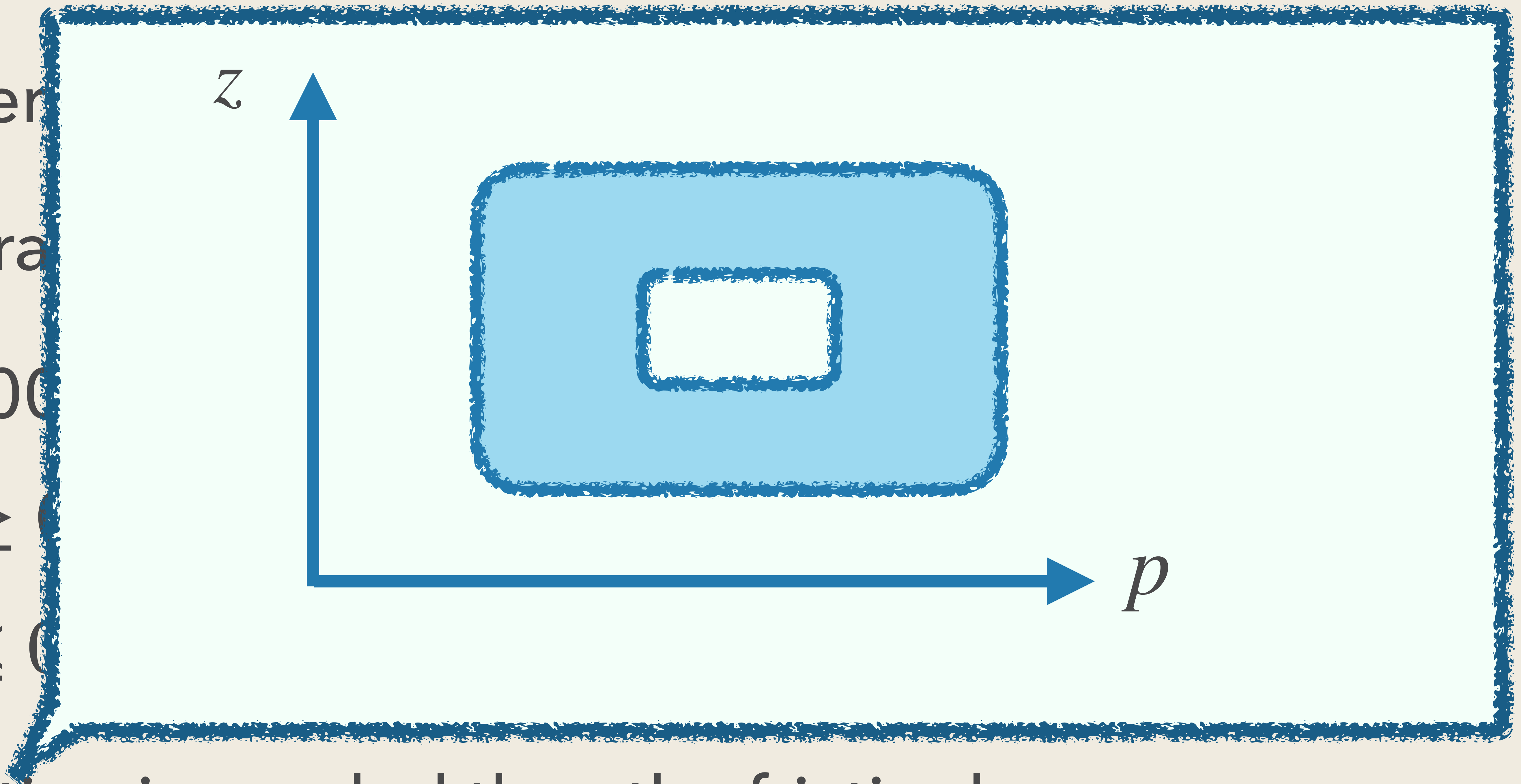


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# Parametric Assumption

- Suppose

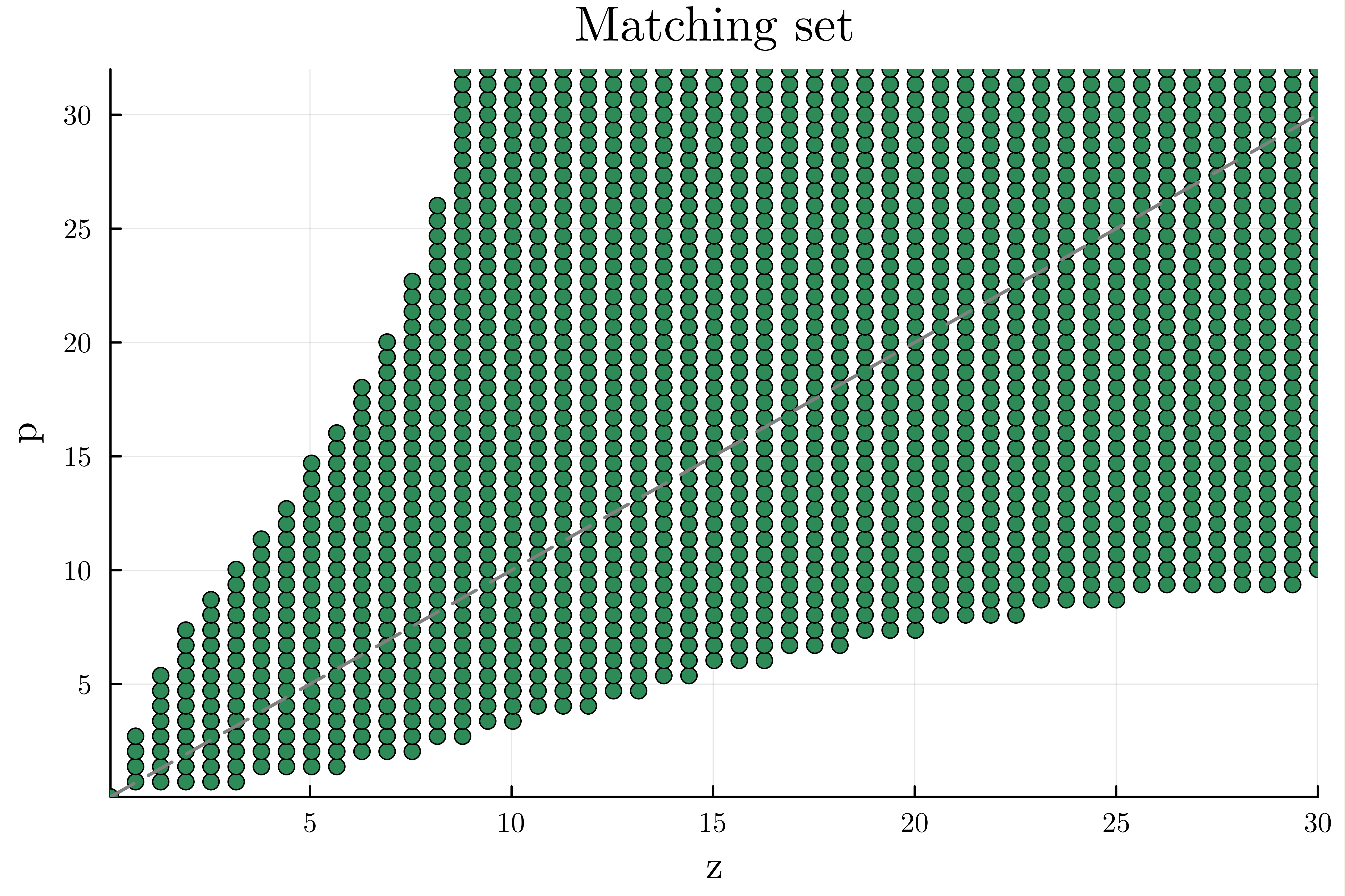
$$f(z,p) = \left(\alpha z^{\frac{\varsigma-1}{\varsigma}} + (1-\alpha)p^{\frac{\varsigma-1}{\varsigma}}\right)^{\frac{\varsigma}{\varsigma-1}}$$

  - $\varsigma < 1 \Rightarrow$ PAM

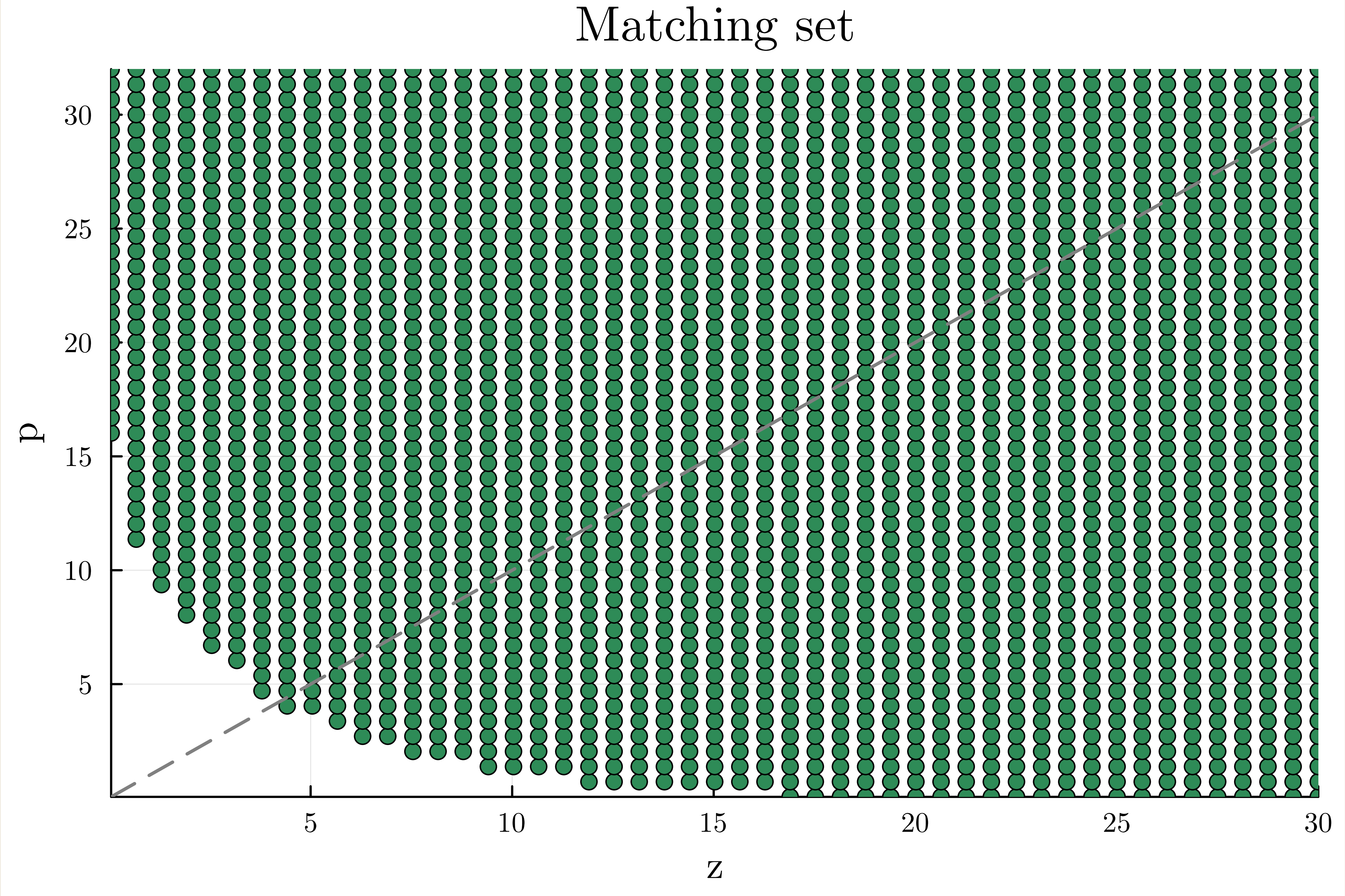
- No theoretical result, but numerically, sufficiently high $\varsigma \Rightarrow$ NAM

# Matching Sets

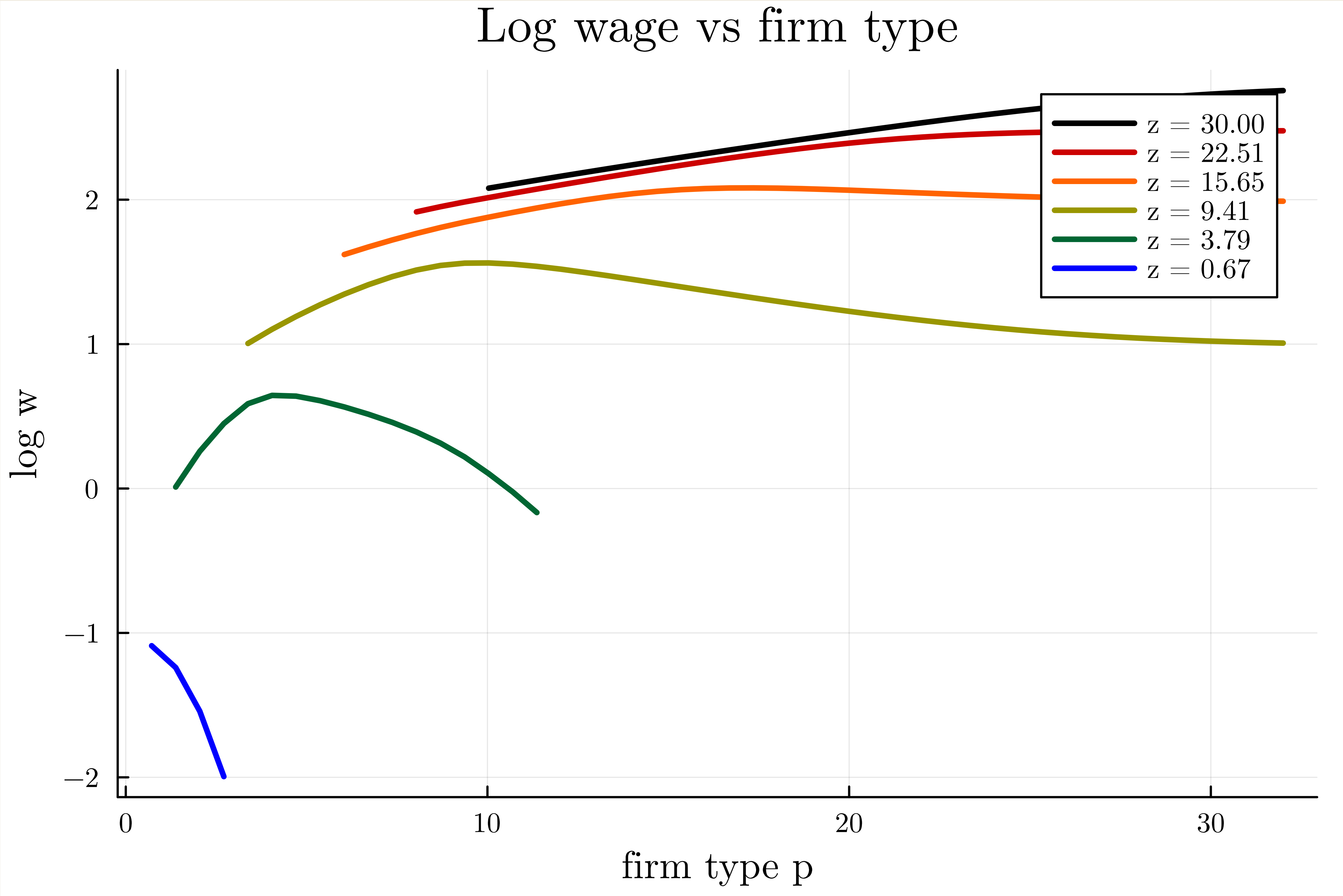
$\varsigma = 0.1$

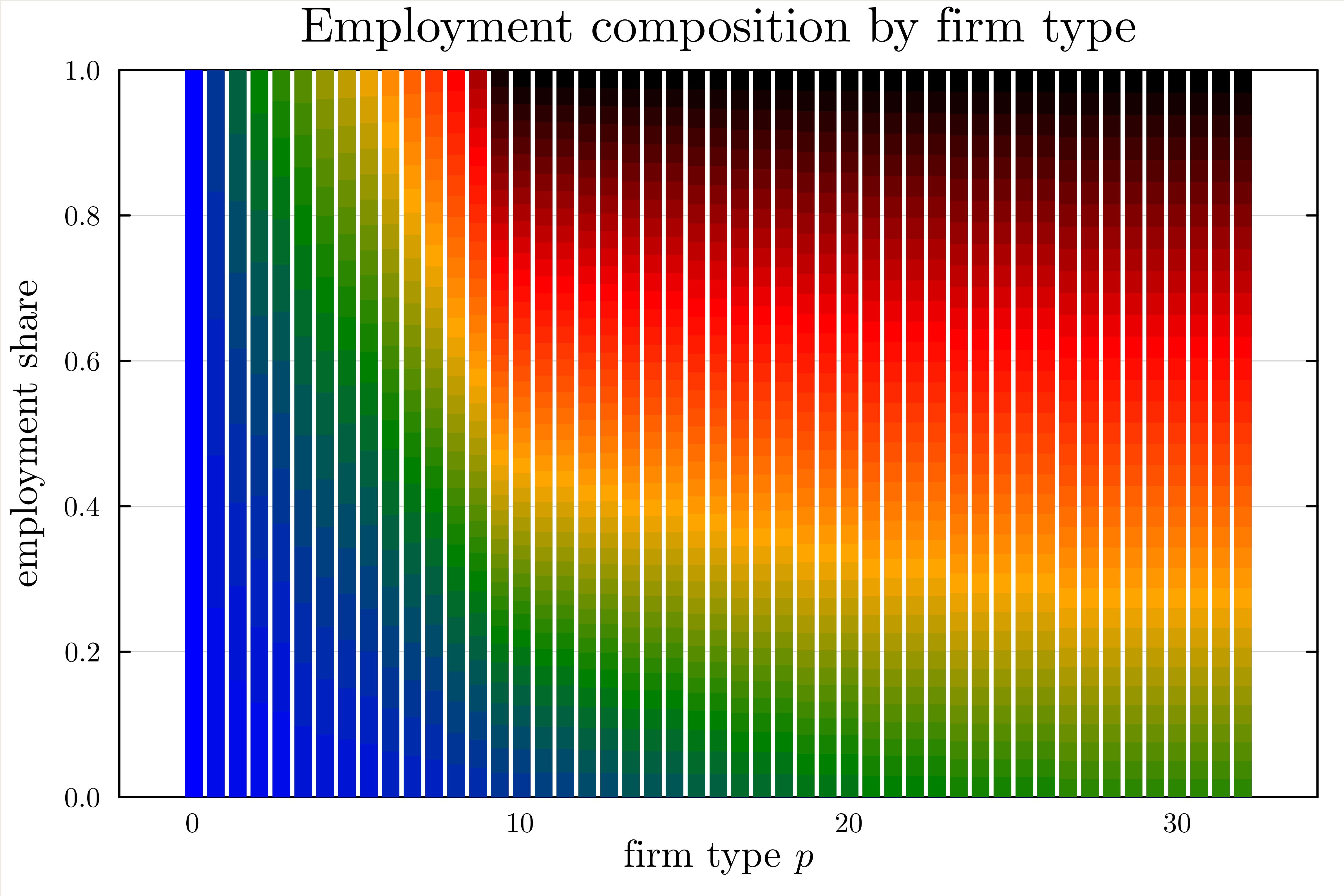



$\varsigma = 3$

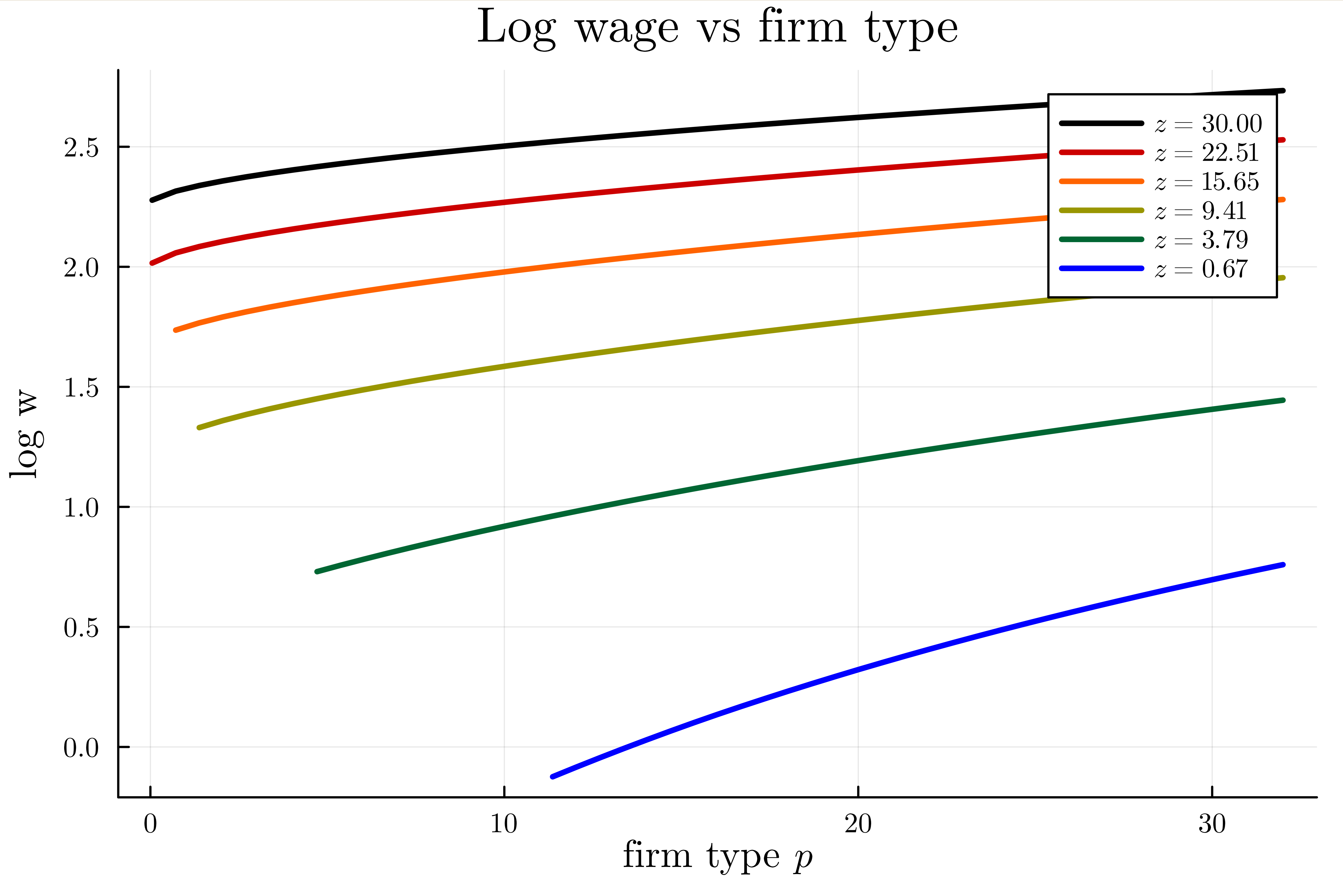


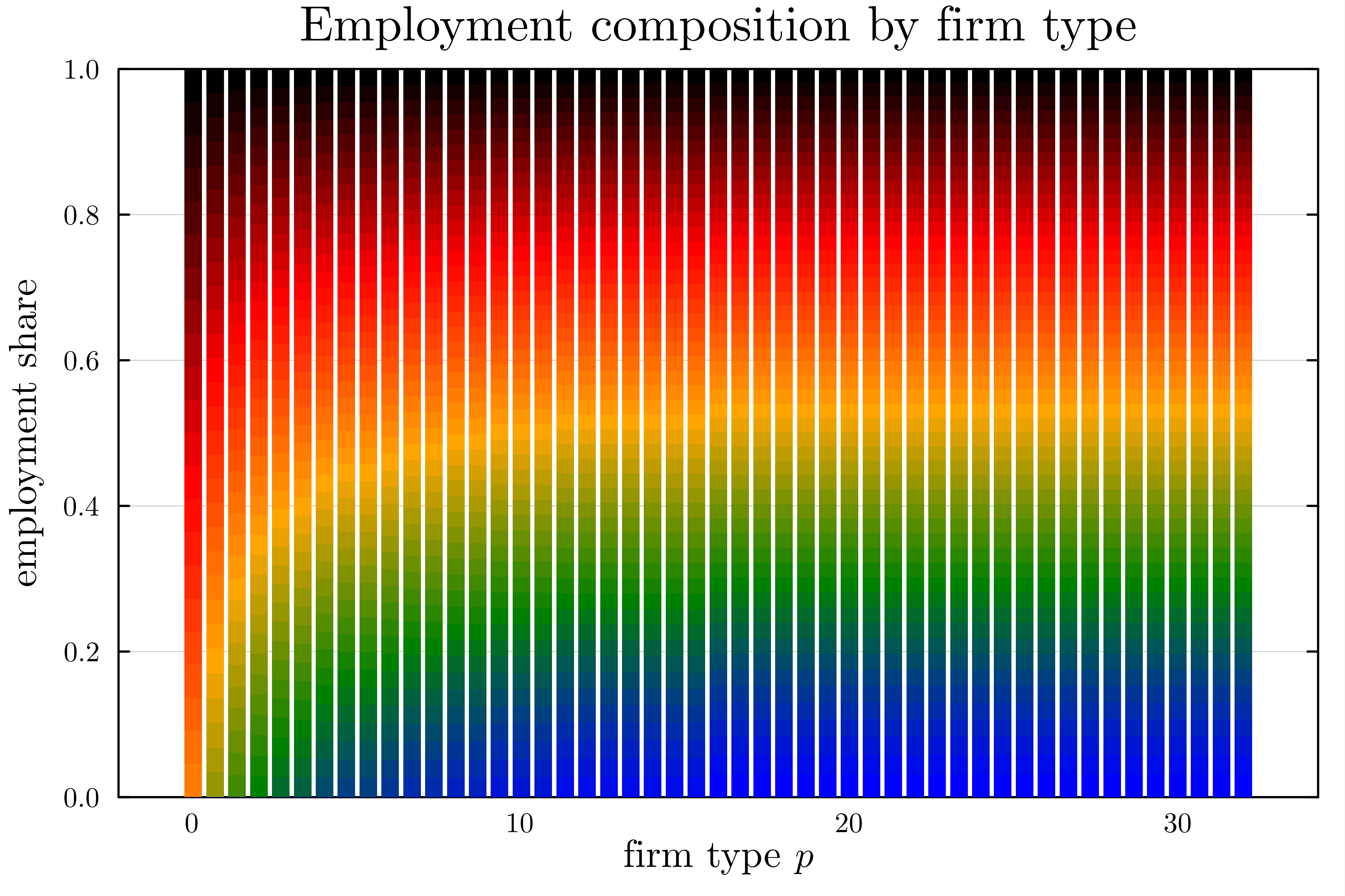
# PAM ($\varsigma = 0.1$)



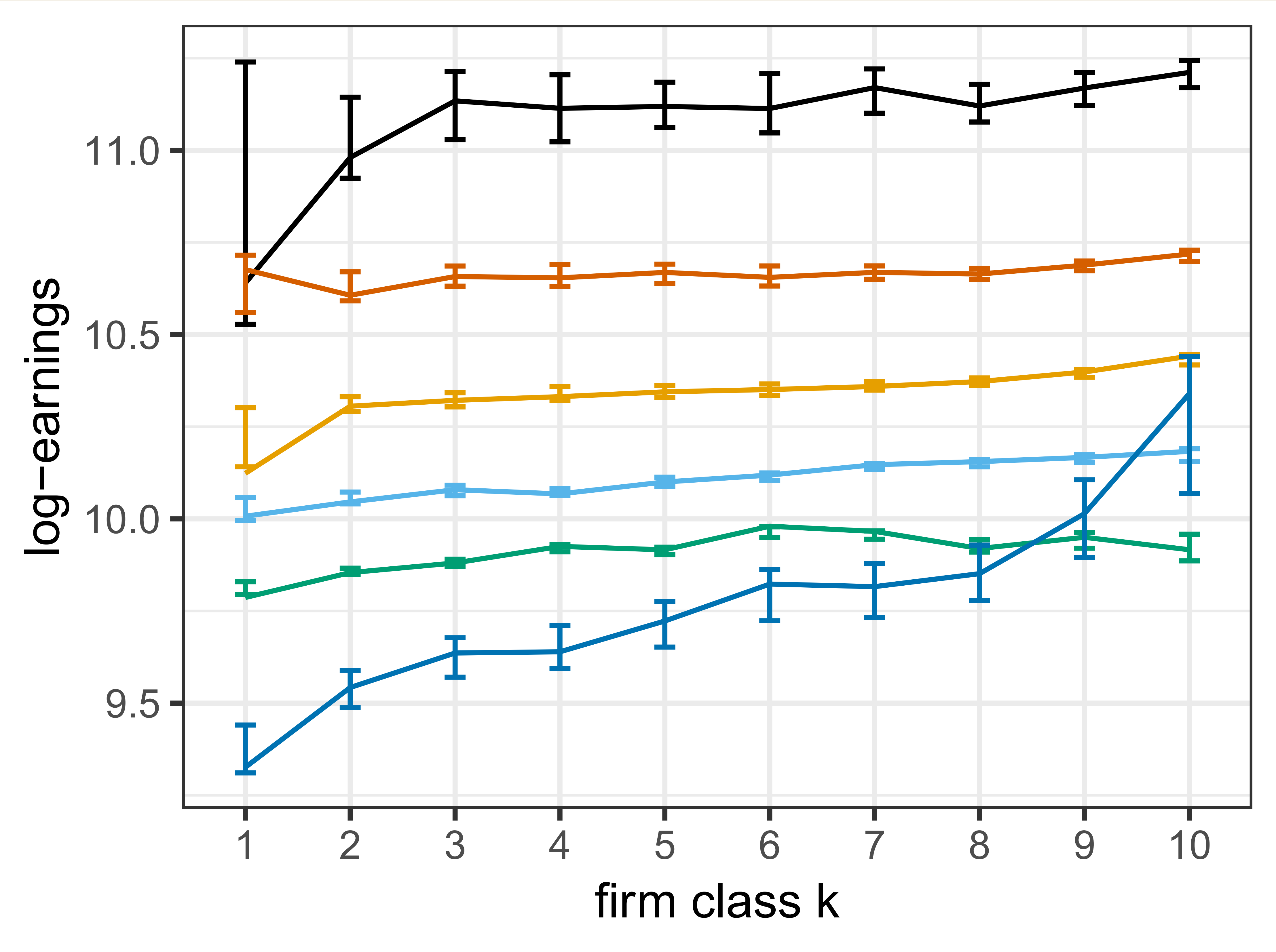



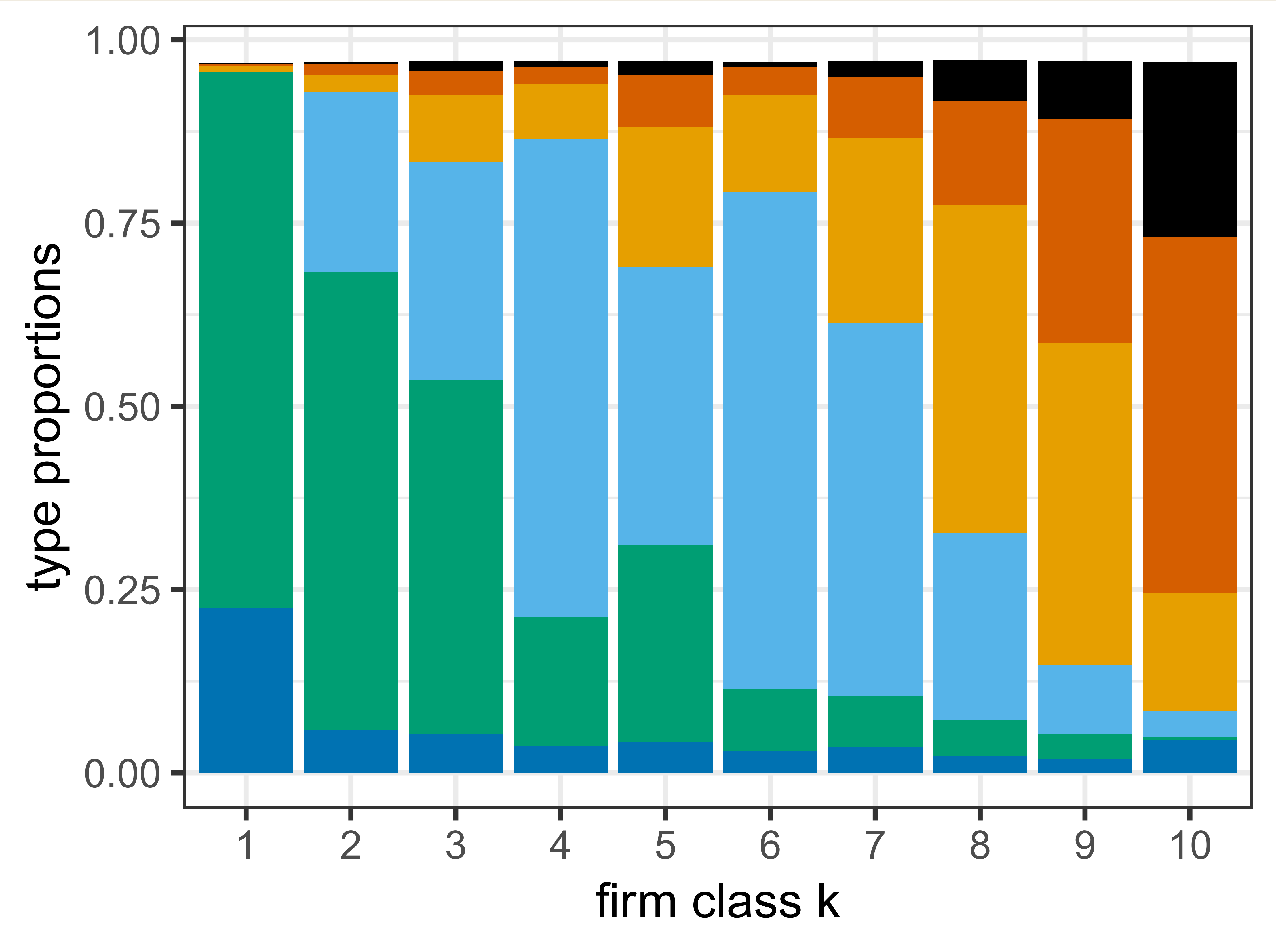
# NAM ($\varsigma = 3$)



# Recap: Data in BLM (2019)



# Puzzle

*"the presence of strong sorting, together with the absence of strong complementarities in wages, is difficult to reconcile with models where sorting is driven by complementarities in production"*

— Bonhomme, Lamadon and Manresa (2019)

# Sorting and Wages with Selection

## – Borovičková & Shimer (2025)

# Solution to the Puzzle

- Extend Shimer & Smith (2000) with idiosyncratic match quality
- When a worker and a firm meet, they draw a match quality $\omega \sim G(\omega)$
- They then decide whether to form a match or not
- A match $(z, p)$ with match quality $\omega$ produces $\omega f(z, p)$
- This is the only modification

# Value Functions

- Value functions:

$$\rho U_z = \chi_0 \gamma \sum_p v_p \int_0^\infty \max\{S_{z,p}(\omega), 0\} dG(\omega)$$

$$\rho V_p = \chi_0 (1-\gamma) \sum_z u_z \int_0^\infty \max\{S_{z,p}(\omega), 0\} dG(\omega)$$

$$(\rho + \delta) S_{z,p}(\omega) = \omega f(z, p) - \rho U_z - \rho V_p$$

- The employment value is (in order to think about wage)

$$\rho W_{z,p}(\omega) = w_{z,p}(\omega) + \delta(U_z - W_{z,p}(\omega))$$

# Steady State Distribution

- The steady state distribution satisfies

$$\delta\phi_{z,p} = \chi_0 u_z v_p (1 - G(\underline{\omega}_{z,p}))$$

$$u_z = m_z - \sum_p \phi_{z,p}$$

$$v_p = n_p - \sum_z \phi_{z,p}$$

# Equilibrium Analysis

- The reservation match quality above which the match is formed, $\underline{\omega}_{z,p}$, satisfies

$$\underline{\omega}_{z,p} = \frac{\rho U_z + \rho V_p}{f(z,p)}$$

- The wage is given by

$$w_{z,p}(\omega) = \rho U_z + \gamma \left[ \omega f(z,p) - \rho U_z - \rho V_p \right]$$

- Combining the above two expressions,

$$w_{z,p}(\omega) = \rho U_z + \gamma(\rho U_z + \rho V_p) \left[ \frac{\omega}{\underline{\omega}_{z,p}} - 1 \right]$$

# Exact AKM in Level

Suppose $G$ is a Pareto, $G(\omega) = 1 - (\omega/\breve{\omega})^{-\theta}$ with sufficiently low $\breve{\omega}$.

Then, the wage of worker of $i$ of type $z_i$ employed at firm $j$ of type $p_j$ is

$$w_{i,j} = \alpha_{z_i} + \psi_{p_j} + \epsilon_{i,j}$$

where

$$\alpha_z \equiv \left(1 + \frac{\gamma}{\theta - 1}\right)\rho U_z, \quad \psi_p \equiv \frac{\gamma}{\theta - 1}\rho V_p, \quad \epsilon_{i,j} \equiv \gamma(\rho U_z + \rho V_p)\left(\frac{\omega}{\underline{\omega}_{z,p}} - \frac{\theta}{\theta - 1}\right)$$

and $\mathbb{E}[\epsilon_{i,j} | z = z_i, p = p_j] = 0$.

- This is an exact AKM equation in **level**!

# Proof

- Manipulating the wage equation,

$$w_{z,p}(\omega) = \rho U_z + \gamma(\rho U_z + \rho V_p)\left[\frac{\omega}{\underline{\omega}_{z,p}} - 1\right]$$

$$= \rho U_z + \gamma(\rho U_z + \rho V_p)\left[\underbrace{\frac{\omega}{\underline{\omega}_{z,p}} - \frac{\theta}{\theta - 1}}_{\equiv \epsilon_{z,p}(\omega)} + \frac{\theta}{\theta - 1} - 1\right]$$

$$= \rho U_z\left(1 + \frac{1}{\theta - 1}\gamma\right) + \frac{1}{\theta - 1}\gamma\rho V_p + \gamma(\rho U_z + \rho V_p)\epsilon_{z,p}(\omega)$$

- Since $\omega \,|\, \omega \geq \underline{\omega}_{z,p}$ follows Pareto with shape $\theta$ and scale $\underline{\omega}_{z,p}$, we have

$$\mathbb{E}[\epsilon_{z,p}(\omega) \,|\, z, p] = 0$$

# Unpacking Striking Implications

$$w_{i,j} = \alpha_{z_i} + \psi_{p_j} + \epsilon_{i,j}$$

The previous result is striking in many ways

- The goods news is that it provides a structural interpretation of AKM but in **level**
  - In practice, whether to take a log or not matters little, so good news
  - Unlike Morchio-Moser (2025), we have an error term satisfying $\mathbb{E}_{z_i,p_j}[\epsilon_{i,j}] = 0$
- The bad news is that wages are useless to learn about $f(z, p)$
- Regardless of $f(z, p)$:
  - High $V_p$ firms pay higher wages for any worker on average
    - In contrast to Shimer-Smith!
  - High $U_z$ firms earn higher wages at any firm

# Sorting

- What about sorting?
- We say $\phi_{z,p}$ satisfies the monotone likelihood ratio property (MLRP) if

$$\frac{\phi_{z_2,p_2}}{\phi_{z_1,p_2}} > \frac{\phi_{z_2,p_1}}{\phi_{z_1,p_1}}$$

  for all $z_1 < z_2$ and $p_1 < p_2$

- The definition states that we are more likely to find "good" workers at "good" firms
- It is equivalent to say $\phi_{z,p}$ is strictly log-supermodular in $(z, p)$ (i.e., $\partial_{z,p}[\ln \phi_{z,p}] > 0$)
- This is an analogue of positive assortative matching in Shimer-Smith

# Positive Assortative Matching

- Under a weak condition on $f(z, p)$, we find positive assortative matching

Suppose $G$ is a Pareto, $G(\omega) = 1 - (\omega/\breve{\omega})^{-\theta}$ with sufficiently low $\breve{\omega}$.

Also assume $f(z, p)$ is strictly increasing and weakly log-supermodular in $(z, p)$ (i.e., $\partial_z f > 0, \partial_p f > 0$, and $\partial_{z,p}[\log f] \geq 0$).

Then, $\phi_{z,p}$ has a monotone likelihood ratio property.

- Despite wage *not* being supermodular, positive assortative matching obtains!
- Provides a unified explanation of the wages and sorting patterns in BLM

# Proof

1. Since $\underline{\omega}_{z,p} = \frac{\rho U_z + \rho V_p}{f(z,p)}$,

$$\partial_{z,p} \ln \underline{\omega}_{z,p} = -\frac{\partial_z U_z \, \partial_p V_p}{(U_z + V_p)^2} - \partial_{z,p} \ln f(z,p) < 0$$

   where we used $U_z$ and $V_p$ are strictly increasing and $f$ is log-supermodular.

2. Using $\delta\phi_{z,p} = \chi_0 u_z v_p (1 - G(\underline{\omega}_{z,p}))$ with Pareto distribution,

$$\begin{aligned} \partial_{z,p}[\ln \phi_{z,p}] &= \partial_{z,p}[\ln(1 - G(\underline{\omega}_{z,p}))] \\ &= -\theta \partial_{z,p}[\ln \underline{\omega}_{z,p}] \\ &> 0 \end{aligned}$$

   which is equivalent to MLRP of $\phi_{z,p}$

# The Role of Selection: ITT and ATT

- Suppose a policymaker wants to increase the wages of worker $z$ employed at $p$
- What if we separate the match and create a meeting between worker $z$ and firm $p'$?
- A naive policymaker, based on AKM, might conclude that it increases wage by

$$\mathbb{E}_{\omega \geq \underline{\omega}_{z,p'}}[w_{z,p'}] - \mathbb{E}_{\omega \geq \underline{\omega}_{z,p}}[w_{z,p}] = \psi_{p'} - \psi_p$$

  - In fact, this is the average treatment effect on treated (ATT)

- Such a conclusion is misleading in this model because the intent to treat (ITT) is

$$\mathbb{E}[w_{z,p'}] - \mathbb{E}_{\omega \geq \underline{\omega}_{z,p}}[w_{z,p}] = \left(1 - G\left(\underline{\omega}_{z,p'}\right)\right) \psi_{p'} - \psi_p$$

  - may well decrease income even if $\psi_{z,p'} > \psi_{z,p}$

# Sorting and Wages with Selection: Quantification with On-the-Job Search

– Borovičková & Shimer (2024)

# Quantitative Model

- Now we would like to see whether the model can quantitatively replicate BLM
- In doing so, we introduce the on-the-job search
  - On-the-job search is how a typical worker switch firms in the data
  - It helps us to match the wage dispersion
- At rate $\chi_1 v_p$, an employed worker meets with firm $p$
- At rate $\chi_1 \phi_{z,p}$, a vacancy meets with an employed worker $z$ at firm $p$
- Assume Nash bargaining with outside option being unemployment
  - no sequential auction

# Worker Value Functions

- Unemployed:

$$\rho U_z = \chi_0 \sum_p v_p \int_0^\infty \max\{W_{z,p}(\omega), 0\} dG(\omega)$$

- Employed:

$$\rho W_{z,p}(\omega) = w_{z,p}(\omega) + \delta(U_z - W_{z,p}(\omega)$$
$$+\chi_1 \sum_{p'} v_{p'} \int \max\{W_{z,p'}(\omega') - W_{z,p}(\omega), 0\} dG(\omega')$$

# Firm Value Functions

- Vacant job:

$$\rho V_p = \chi_0 \sum_z u_z \int_0^\infty \max\{J_{z,p}(\omega), 0\} dG(\omega)$$

$$+\chi_1 \sum_z \sum_{p'} \int \phi_{z,p}(\omega') \int (J_{z,p}(\omega) - V_p) \mathbb{1}_{W_{z,p}(\omega) > W_{z,p'}(\omega')} dG(\omega) d\omega'$$

- Filled job:

$$\rho J_{z,p}(\omega) = \omega f(z,p) - w_{z,p}(\omega) + \delta(V_p - J_{z,p}(\omega))$$

$$+\chi_1 \sum_{p'} v_{p'} \int \mathbb{1}_{W_{z,p'}(\omega') > W_{z,p}(\omega)} dG(\omega')(V_p - J_{z,p}(\omega))$$

# Nash Bargaining

- The wages are determined by Nash bargaining:

$$\max_{w_{z,p}(\omega)} (W_{z,p}(\omega) - U_z)^{\gamma}(J_{z,p}(\omega) - V_p)^{1-\gamma}$$

  - Again, no sequential auction here

- This gives:

$$W_{z,p}(\omega) = U_z + \gamma S_{z,p}(\omega)$$

$$J_{z,p}(\omega) = V_p + (1-\gamma)S_{z,p}(\omega)$$

# Match Surplus

- Substituting the Nash bargaining equations into the value functions,

$$\rho U_z = \chi_0 \gamma \sum_p v_p \int_0^\infty \max\{S_{z,p}(\omega), 0\} dG(\omega)$$

$$\rho V_p = \chi_0 (1-\gamma) \sum_z u_z \int_0^\infty \max\{S_{z,p}(\omega), 0\} dG(\omega)$$

where the joint match surplus $S_{z,p}(\omega)$ solves

$$(\rho + \delta) S_{z,p}(\omega) = \omega f(z,p) - \rho U_z - \rho V_p + \chi_1 \sum_p v_p \int_{S_{z,p'}(\omega') > S_{z,p}(\omega)} \left[\gamma S_{z,p'}(\omega') - S_{z,p}(\omega)\right] dG(\omega')$$

# Steady State Distribution

- Steady state employment distribution satisfies:

$$\phi_{z,p}(\omega)\left(\delta + \chi_1 \sum_{p'} v_{p'} \int_0^\infty \mathbb{1}_{S_{z,p'}(\omega') \geq S_{z,p}(\omega)} dG(\omega')\right)$$

$$= v_p g(\omega)\left(\chi_0 u_z \mathbb{1}_{S_{z,p}(\omega)\geq 0} + \chi_1 \sum_{p'=1}^{Y} \int_0^\infty \mathbb{1}_{S_{z,p}(z)\geq S_{z,p'}(\omega')} \phi_{z,p'}(\omega') d\omega'\right)$$

- The mass of unemployed and vacant jobs are:

$$u_z = m_z - \sum_p \int \phi_{z,p}(\omega) d\omega$$

$$v_p = n_p - \sum_z \int \phi_{z,p}(\omega) d\omega$$

# Calibration

- Annual calibration: $\rho = 0.05$, $\delta = 0.25$, $\gamma = 0.5$, $\chi_1 = 0.2\chi_0$
- 10 worker types, 10 firm types, with each type measure $0.1$
  - $\log z \in [0, \log(1+\Delta_z), 2\log(1+\Delta_z), \ldots, 9\log(1+\Delta_z)]$
  - $\log p \in [0, \log(1+\Delta_p), 2\log(1+\Delta_p), \ldots, 9\log(1+\Delta_p)]$
- Assume

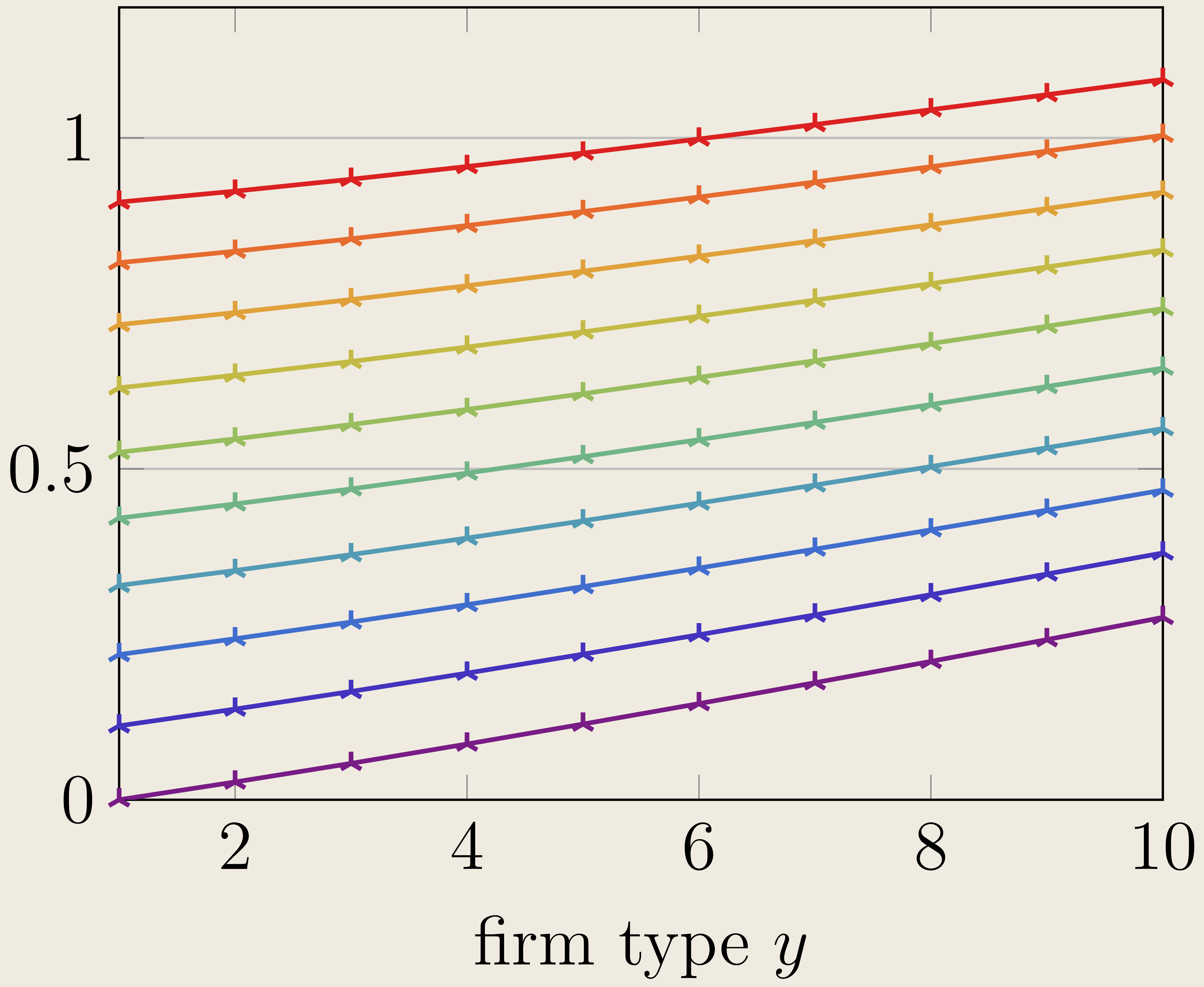
$$f(z,p) = \left(\alpha z^{\frac{\varsigma-1}{\varsigma}} + (1-\alpha)p^{\frac{\varsigma-1}{\varsigma}}\right)^{\frac{\varsigma}{\varsigma-1}} \quad \text{with} \quad \alpha = 1/2$$

- Estimate $(\Delta_z, \Delta_w, \varsigma, \theta)$ to target the wage decomposition in BLM:

| | $\mathrm{var}(\log W)$ | $\mathrm{var}(\alpha)$ | $\mathrm{var}(\psi)$ | $2\mathrm{cov}(\alpha,\psi)$ | $\mathrm{var}(\varepsilon)$ |
|---|---|---|---|---|---|
| BLM | 0.1240 | 0.0747 | 0.0053 | 0.0166 | 0.0274 |

# Successfully Replicate BLM

Average log wage $w^*_{x,y}$, BLM




Share of worker types, BLM