Firm Size Distribution and Firm Dynamics

741 Macroeconomics
Topic 6

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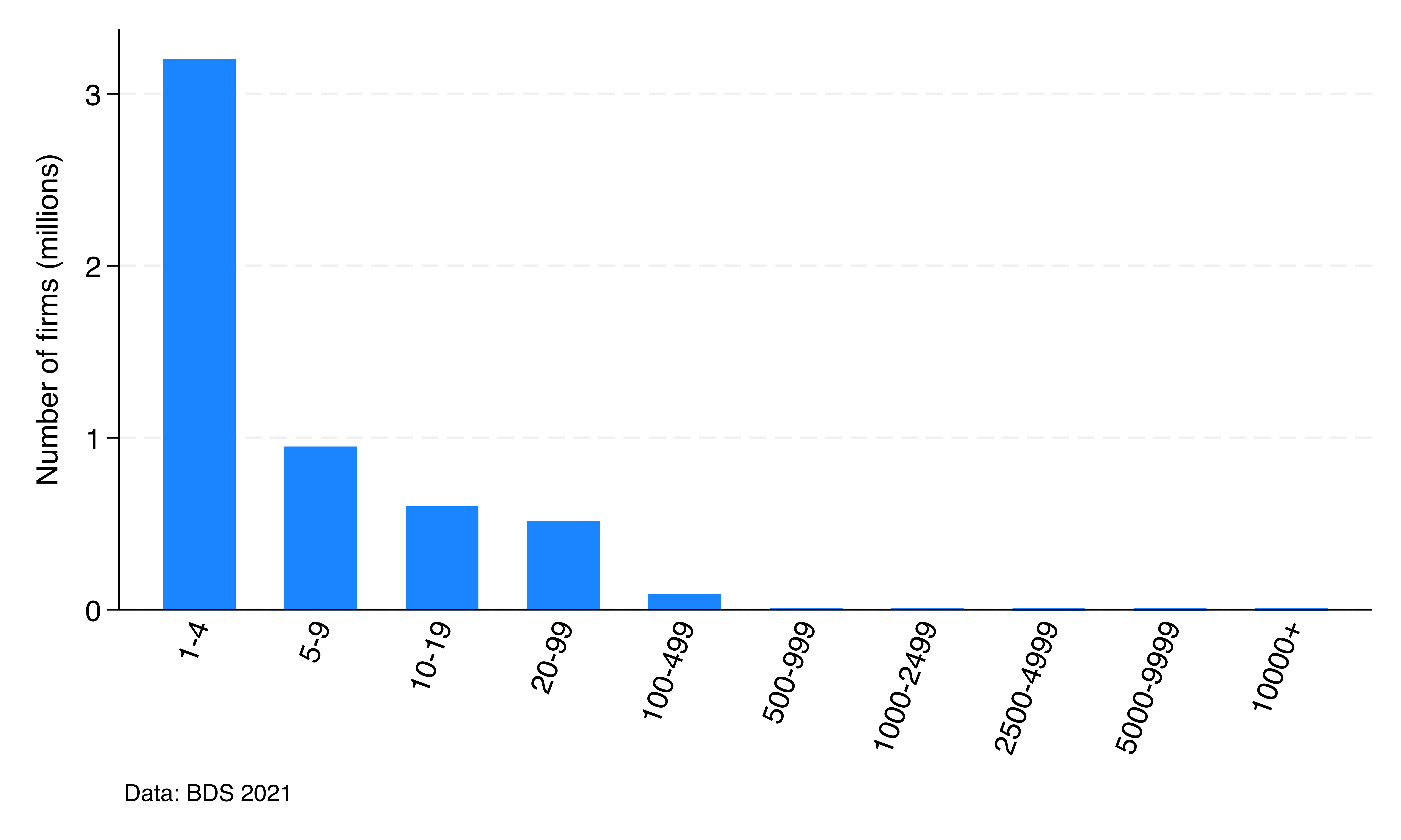
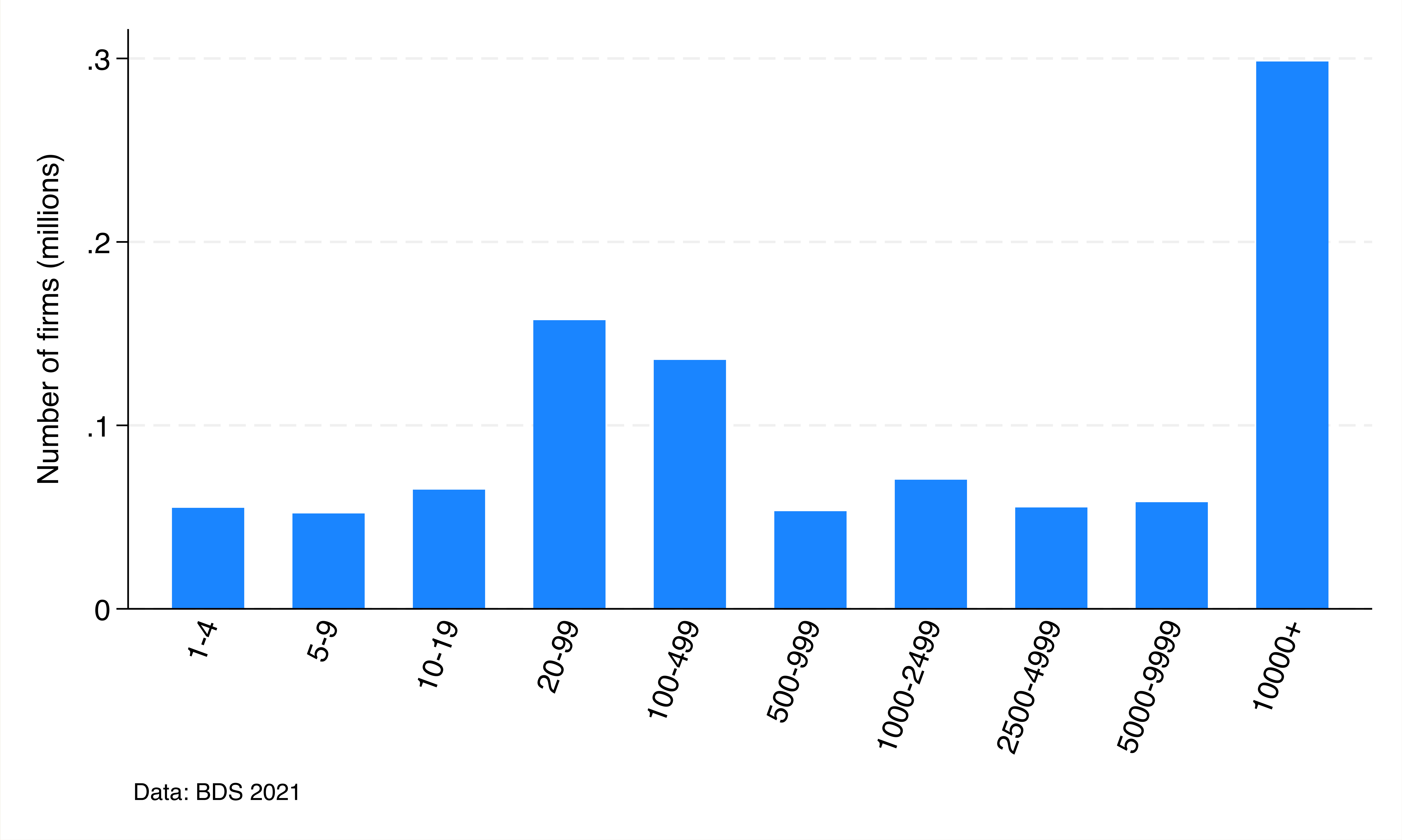
Fall 2025

Firm Size Distribution in the US 2021

Firm Size (Employment) Distribution



Employment Share of Each Size Category

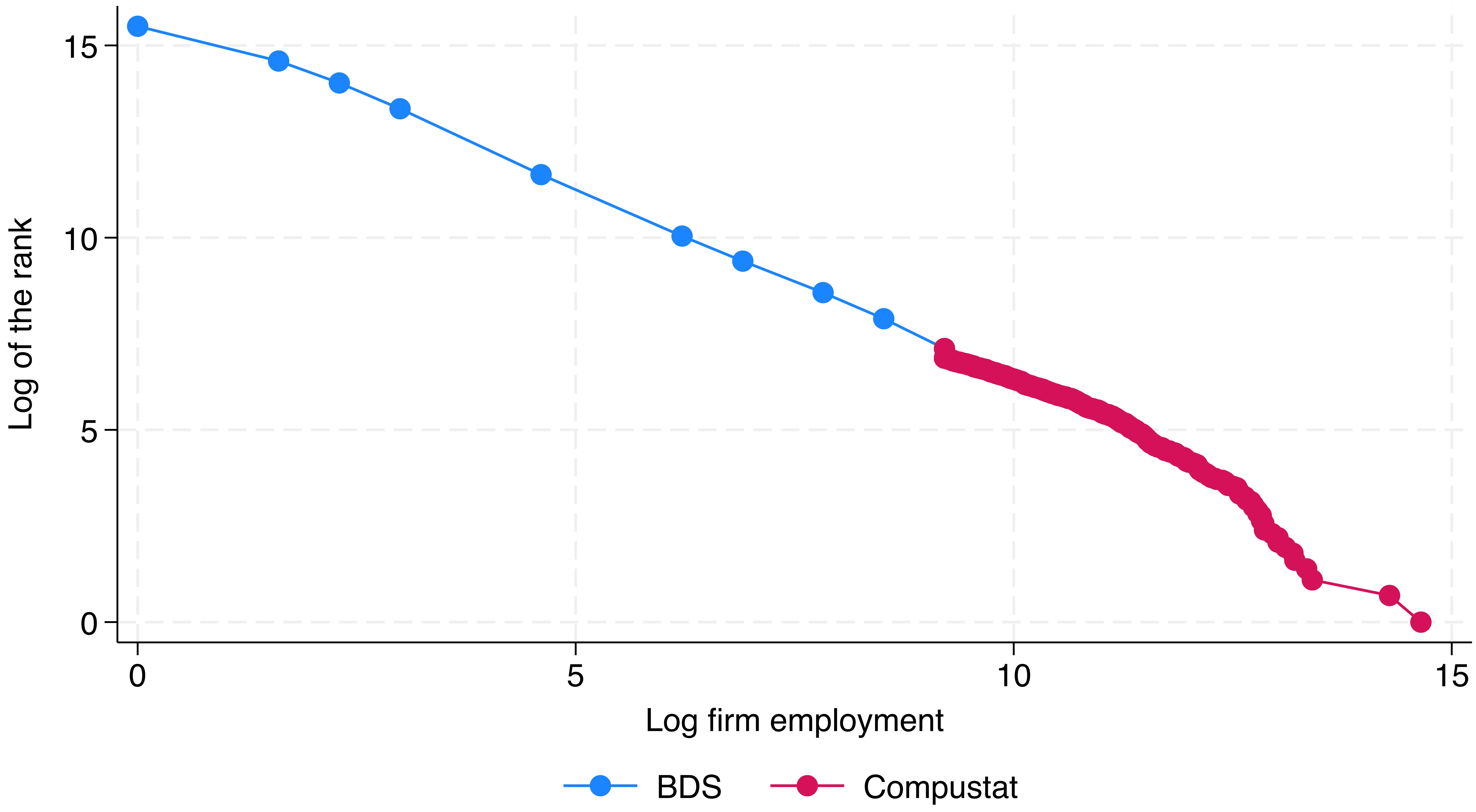


A Handful of Firms Hire Majority of Workers

- Large firms in the US are extremely large
  - Top 0.02% of firms ($\approx$ 1,200 firms) account for 30% of employment in the US
  - Top 1% of firms ($\approx$ 60,000 firms) account for 60% of employment in the US

- What does the right tail of the firm size distribution look like?

Power Law in Firm Size Distribution



Data source: BDS and Compustat 2021

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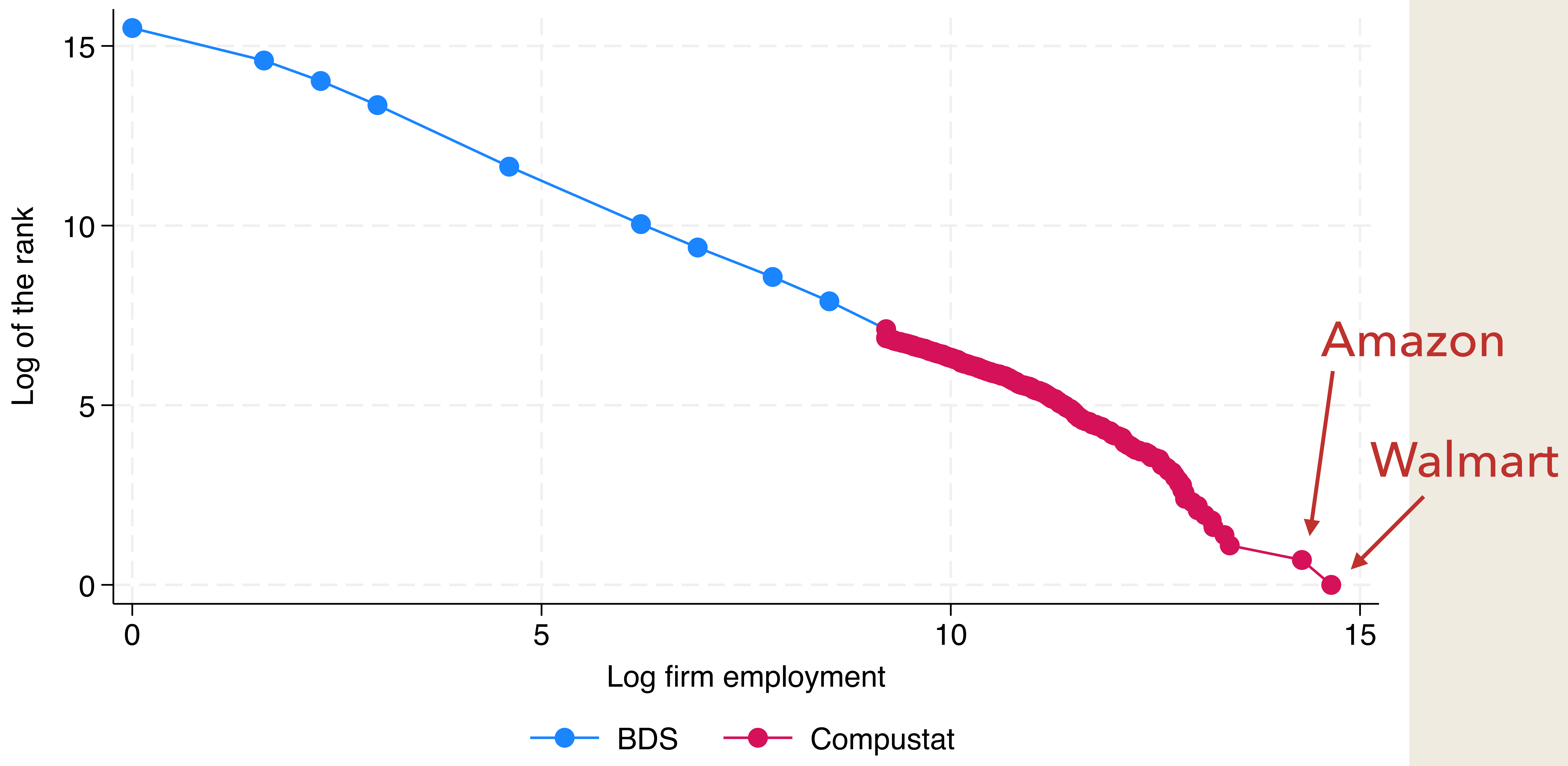

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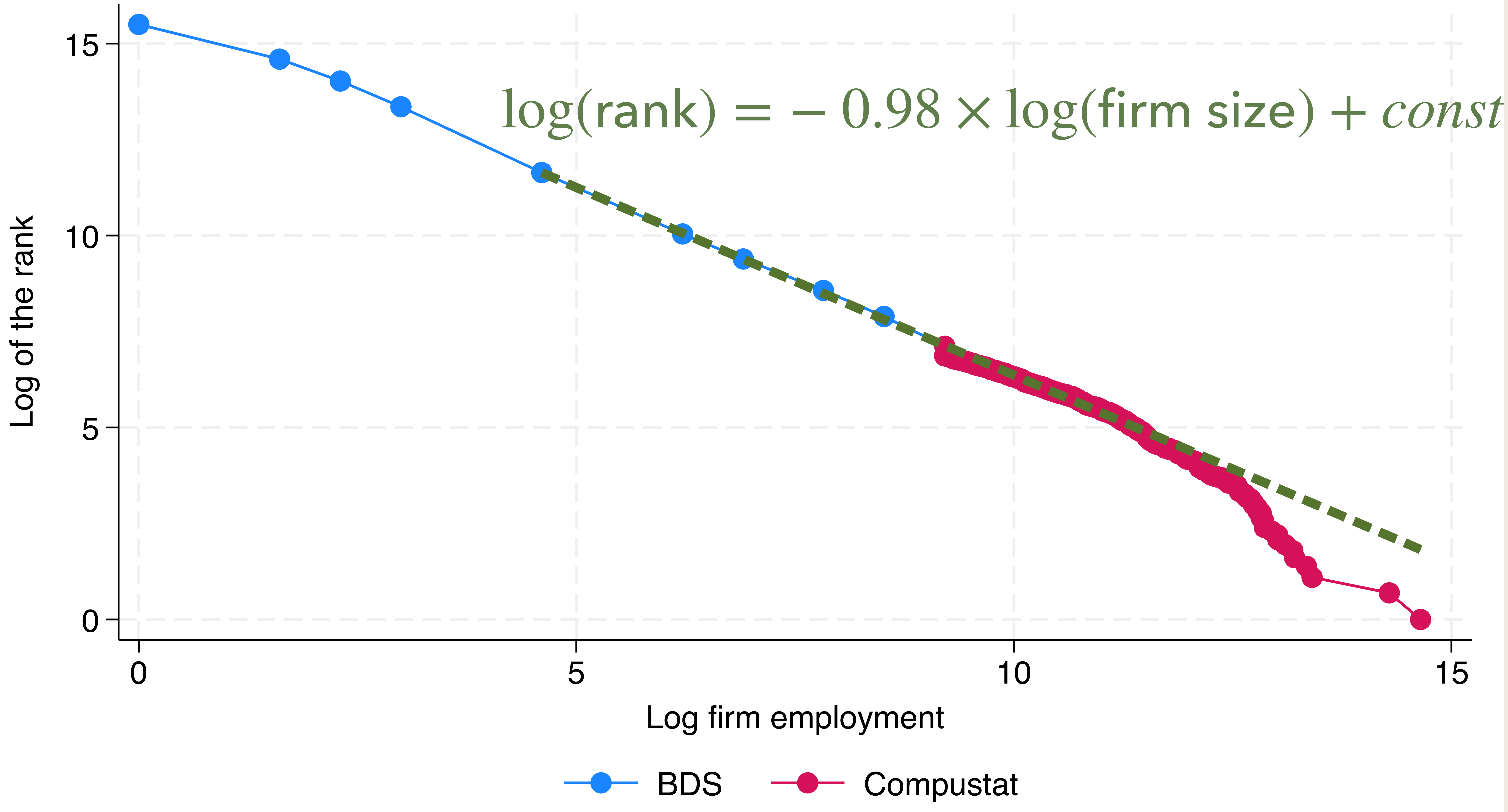

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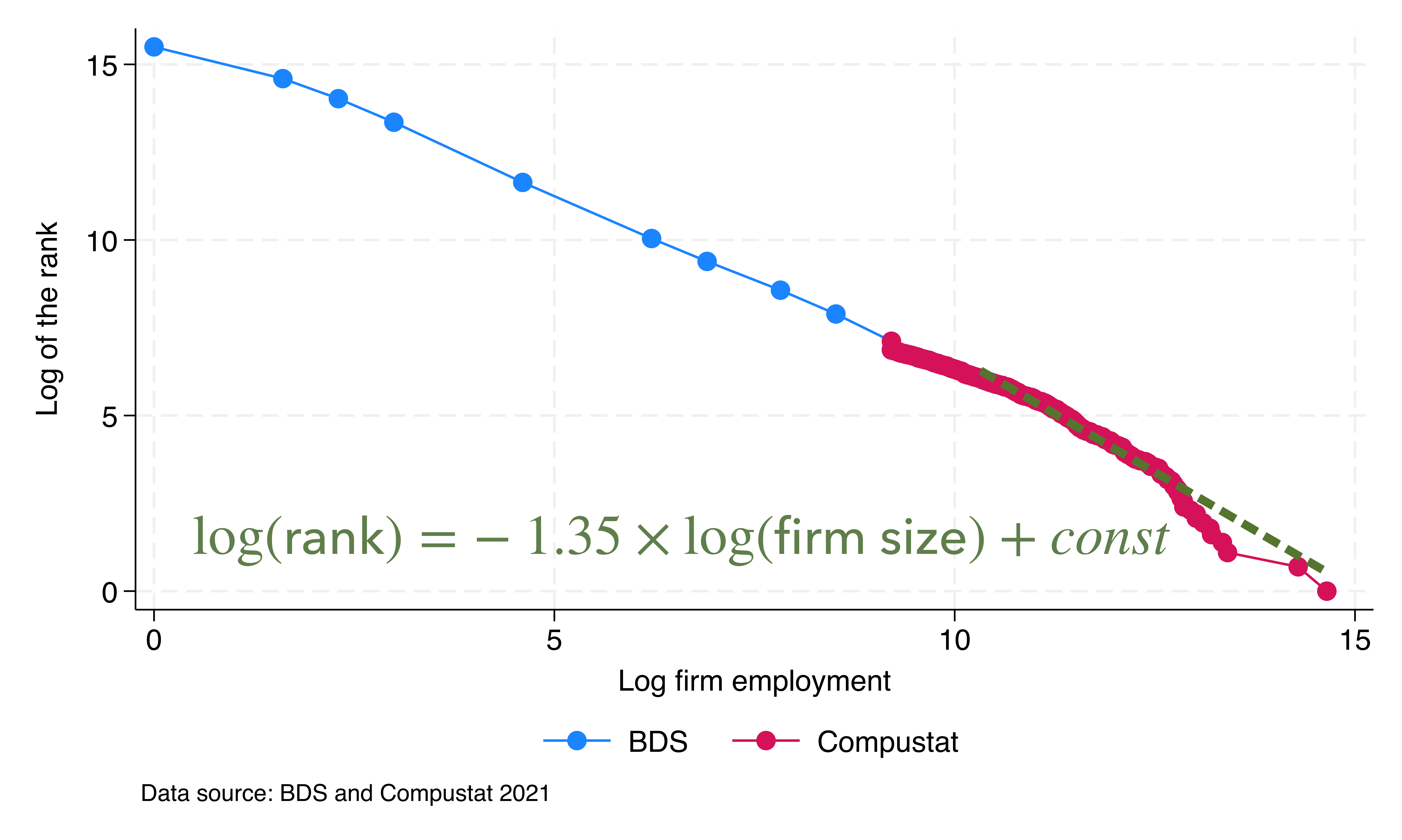


Two Facts in Firm Size Distribution

- Two surprises:
  1. The ranking of firm size is log-linear in firm size (**Power law**)
  2. The coefficient is close to one (**Zipf's law**)
- Mathematically,

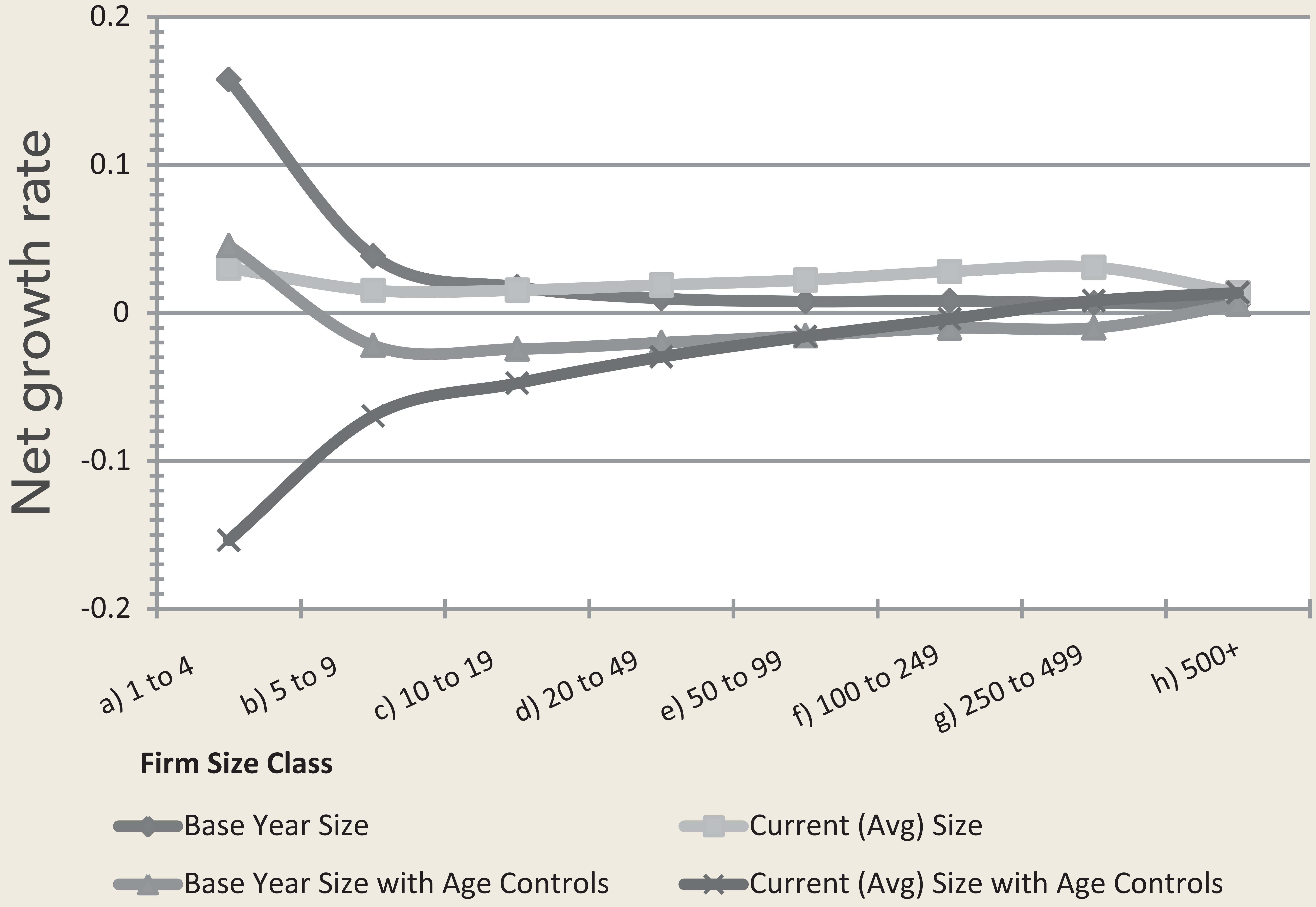
$$\log \underbrace{\mathrm{Pr}(\tilde{x} \geq x)}_{\text{ranking}} = -\zeta \log x + const, \qquad \zeta \approx 1$$

- What is this distribution?
  – Pareto: $\mathrm{Pr}(\tilde{x} \geq x) = (x/\underline{x})^{-\zeta}$

The Nature of Firm Growth

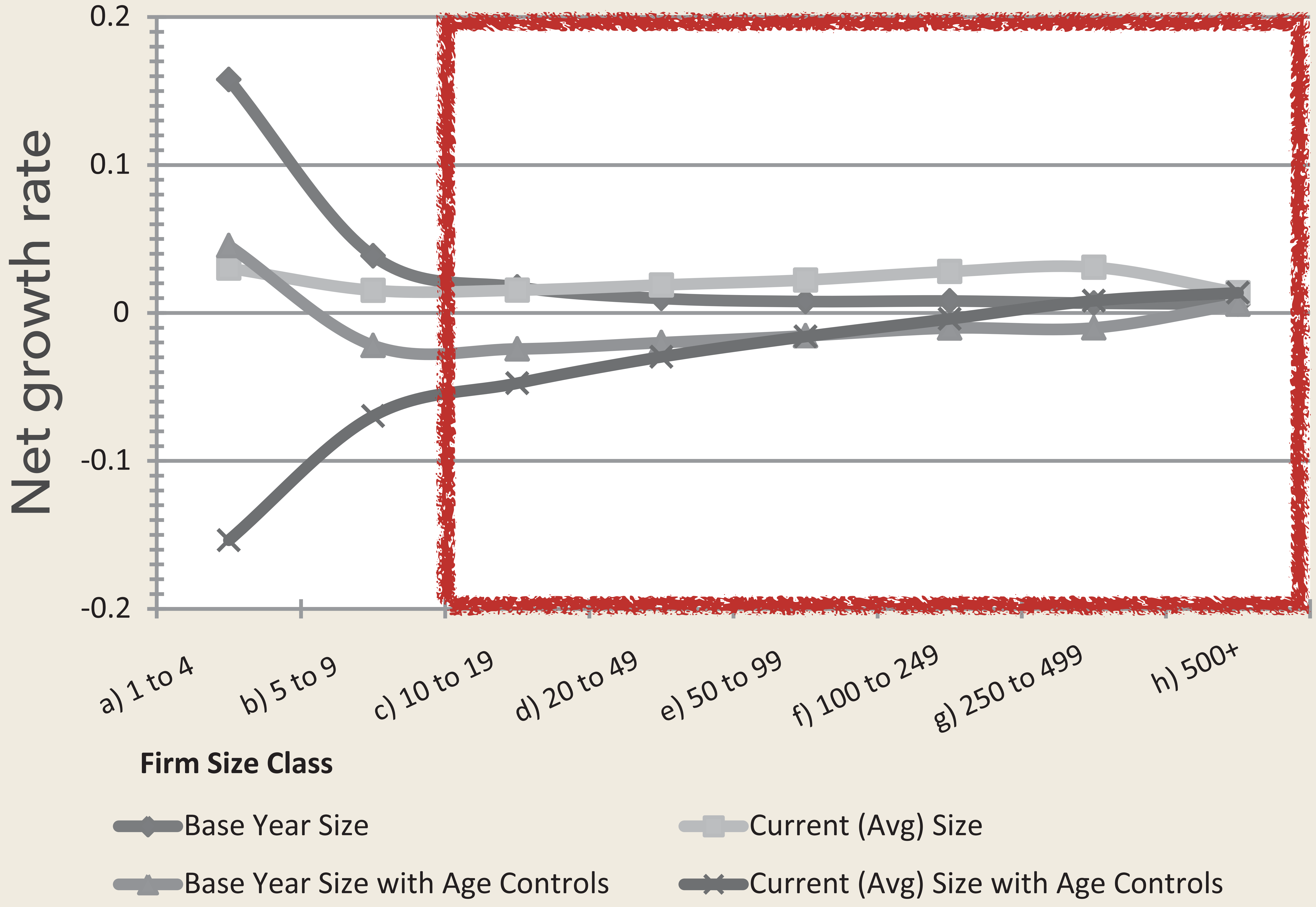
- How do large firms grow going forward?
  - Do they systematically shrink? (i.e., mean reversion in firm size)
  - Do they keep outperforming other smaller firms?
- Look at the relationship between firm growth and firm size

Firm Growth and Firm Size



Source: Haltiwanger, Jarmin & Mirand (2013)

Firm Growth and Firm Size



Source: Haltiwanger, Jarmin & Mirand (2013)

Gibrat's Law

- Firm growth rate is roughly independent of firm size…
  … if we exclude small firms
- This is called Gibrat's law

A Mechanical Model of Firm Size Distribution

Connecting Two Laws

- Two robust features of the firm dynamics
  1. Power law
  2. Gibrat's law

- Gabaix (1999): Gibrat's law $\Rightarrow$ Power law

Random Growth Model

- We consider a mechanical model of firm growth
- Let n_t denote the firm size, which is stochastic
- Gibrat's law suggests random growth:

$$n_{t+1} = \gamma_{t+1} n_t$$

where $\gamma_{t+1} \sim F(\gamma)$ follows some distribution and is independent of firm size n_t

Law of Motion for Distribution

- The counter CDF of the firm size distribution, $\bar{G}_t(n) \equiv \text{Prob}(n_t > n)$, follows

$$
\begin{aligned}
\bar{G}_{t+1}(n) &= \mathsf{P}(n_{t+1} > n) \\
&= \mathsf{P}(\gamma_{t+1} n_t > n) \\
&= \mathbb{E}\left[\mathbf{I}[n_t > n/\gamma_{t+1}]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{I}[n_t > n/\gamma_{t+1}] \,|\, \gamma_{t+1}\right]\right] \\
&= \mathbb{E}\left[\bar{G}_t(n/\gamma_{t+1})\right] \\
&= \int \bar{G}_t(n/\gamma_{t+1}) dF(\gamma)
\end{aligned}
$$

Steady State Distribution is Pareto

- The steady state distribution, $\bar{G}(n)$, if it exists, satisfies

$$\bar{G}(n) = \int \bar{G}(n/\gamma) dF(\gamma) \tag{1}$$

- Is Pareto, $\bar{G}(n) = (n/\underline{n})^{-\zeta}$, the steady state distribution?
- Substittuing $\bar{G}(n) = (n/\underline{n})^{-\zeta}$ into (1) gives

$$\mathbb{E}[\gamma^{\zeta}] = 1$$

so yes, Pareto is a natural candidate solution

- Could it be $\zeta \approx 1$ (Zipf's law)? – Yes if

$$\mathbb{E}[\gamma] \approx 1$$

Intuition

1. Why Gibtrat's law $\Rightarrow$ power law?
   - Random growth implies scale invariance
   - Then the final distribution needs to be scale-invariant $\Rightarrow$ Pareto
2. Why average zero growth $\Rightarrow$ Zipf's law?
   - Suppose the firm size can either double or halve
   - Then, to have a zero growth rate, $P(\text{double}) = 1/3$ and $P(\text{halve}) = 2/3$
     - $2 \times 1/3 + 1/2 \times 2/3 = 1$
   - The number of firms with size $2n$ is half the number of firms with size n $\Rightarrow$ Zipf's law

Existence of Steady State Distribution

- Does the steady state exist?
- Without further assumptions, the answer is no
- To see this,

$$\ln n_{t+1} = \sum_{s=0}^{t} \ln \gamma_{s+1} + \ln n_0$$

so that

$$\text{Var}(\ln n_{t+1}) = \text{Var}(\ln \gamma)t + \text{Var}(\ln n_0)$$

which grows without bound

- However, if there are vanishingly small stabilizing forces, SS distribution exists, e.g.,
  - A minimum size requirement, $n \geq n^{\min}$
  - A stochastic exit

Formal Results

- Suppose $n_{t+1} = \max\{\gamma_{t+1} n_t, n_{\min}\}$, where $\gamma_{t+1} \sim F(\gamma)$
- Then, for some ζ satisfying $\mathbb{E}[\gamma^\zeta] = 1$, we have

$$\frac{1}{n^\zeta} \Pr(\tilde{n} > n) \to const \quad \text{as } n \to \infty$$

  - Firm size distribution is asymptotically Pareto
- In continuous time, the firm size distribution is globally Pareto
  - If interested, see my lecture notes from 2024

Canonical Model of Firm Dynamics

– Hopenhayn and Rogerson (1993)

Environment

- Firms:
  - ex-post heterogeneity in productivity
  - decreasing returns to scale production
  - entry and exit
- Competitive labor market!
- Timing within a period:
  1. Firms enter/exit
  2. Produce & pay wages

Technology

- We assume the firm's production function is

$$f(n,z) = z^{1-\alpha} n^{\alpha}$$

  - z: idiosyncratic productivity, n: employment

- The firm's profit function is

$$\pi(z) = \max_n f(n,z) - wn - c_f$$

  - c_f : fixed cost of operation

- Solutions:

$$n(z) = (\alpha/w)^{\frac{1}{1-\alpha}} z, \quad \pi(z) = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha) w^{\frac{-\alpha}{1-\alpha}} z - c_f$$

  $\Rightarrow$ Firm size n is proportional to firm productivity

- Assume z follows a Markov process

Firm's Exit Decision

- Firms can always exit, in which case the firm obtains zero value
- The firm's problem in recursive form is

$$v(z) = \max\{v^*(z), 0\}$$

where v^* is the continuation value:

$$v^*(z) = \pi(z) + \beta \mathbb{E}\left[v(z') \,|\, z\right]$$

  - $\beta \equiv 1/(1+r)$ is a discount factor, and $v^*(z)$ is the continuation value
- It is easy to show $v^*(z)$ is strictly increasing in z, implying
  - If $z > \underline{z}$, firms continue
  - If $z < \underline{z}$, firms exit

Free Entry

- There is a large mass of potential firms that can create firms with cost c_e
- Upon entry, firms draw z from pdf $\psi_0(z)$
- The free-entry condition is (assuming positive entry):

$$\int v(z)\psi_0(z)dz = c_e$$

Evolution of Distribution

- Let m denote the mass of entrants
- Let $g(z)$ denote the mass of firms with productivity z
- The steady state distribution is given by

$$g(z) = \begin{cases} \int \Pi(z \,|\, z_{-1}) g(z_{-1}) dz_{-1} + m\psi_0(z) & \text{for } z \geq \underline{z} \\ 0 & \text{for } z < \underline{z} \end{cases}$$

Household and Labor Supply

- Assume households have a labor endowment L
- Let $g(z)$ denote the mass of firms with productivity z
- The household's problem is

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t C_t$$

$$\text{s.t.} \quad C_t = wL + \int \pi(z) g(z) dz - mc_e$$

- Labor market clearing is

$$\int n(z) g(z) dz = L$$

Equilibrium Definition

- Equilibrium: $\{v(z), \underline{z}, w\}$ and $\{g(z), m\}$ such that:
  1. Given w, $\{v(z), \underline{z}\}$ solve the Bellman equation
  2. Free entry holds, $\int v(z)\psi_0(z)dz = c_e$
  3. $\{g(z), m\}$ satisfies the steady state condition
  4. Labor market clears

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1 & 2 alone $\Rightarrow \{v(z), \underline{z}, w\}$

- Free entry alone is enough to pin down w
  - Put differently, wages need to adjust so as to ensure free entry

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  - Value & policy functions are independent of distribution or labor supply

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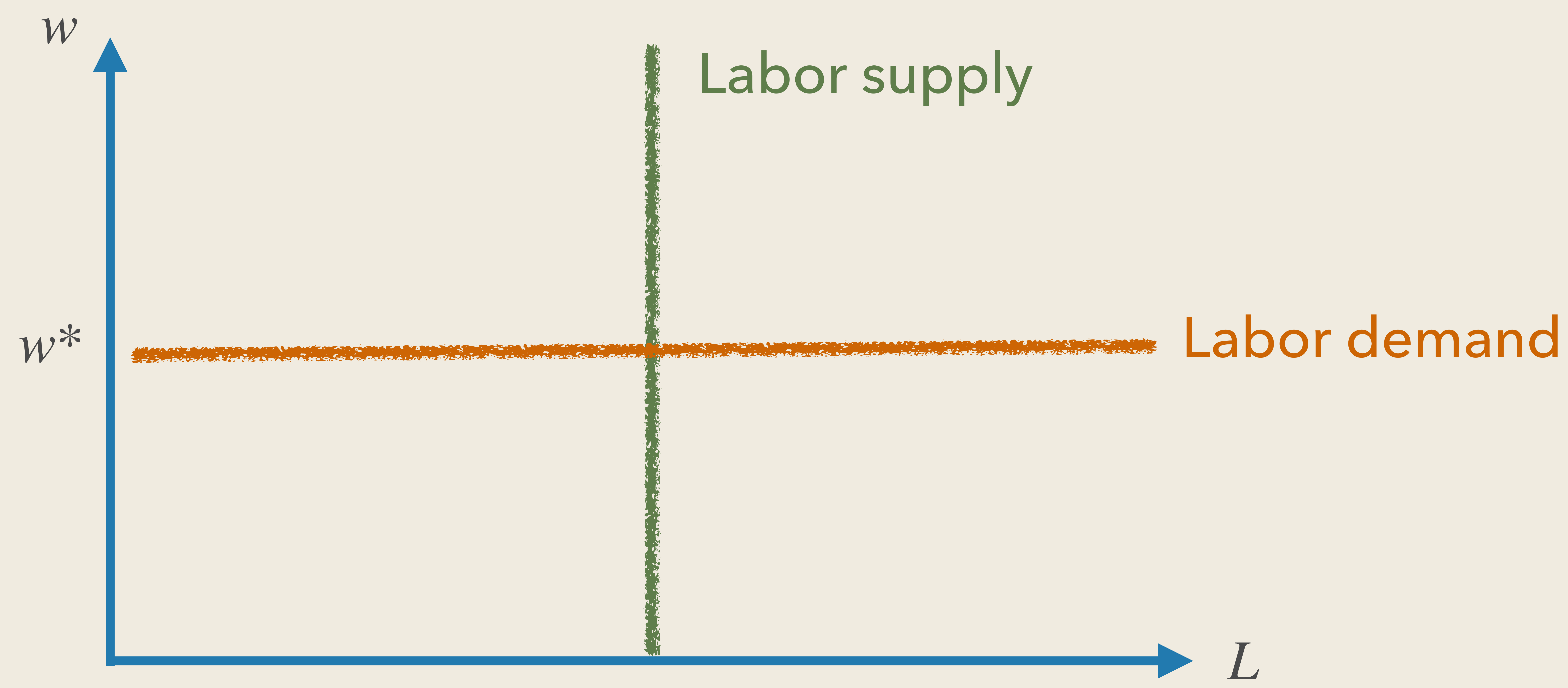
- Free entry alone is enough to pin down w
  - Put differently, wages need to adjust so as to ensure free entry
- This is a **block recursive** property (again)!
  - Value & policy functions are independent of distribution or labor supply
- The mass of entrants m adjusts to clear the labor market

Horizontal Aggregate Labor Demand



- Labor demand horizontal: $w > w^* \Rightarrow$ infinite demand; $w < w^* \Rightarrow$ no demand.
- For a given wage $w = w^*$, the labor market clears because entry adjusts

Distribution Block

- Define $\hat{g}(z) \equiv g(z)/m$ and $\hat{g}(z)$ solves

$$\hat{g}(z) = \begin{cases} \int \Pi(z \,|\, z_{-1}) \hat{g}(z_{-1}) dz_{-1} + \psi_0(z) & \text{for } z \geq \underline{z} \\ 0 & \text{for } z < \underline{z} \end{cases}$$

which we can solve for $\hat{g}(z)$ conditional on the knowledge of $\underline{z}$

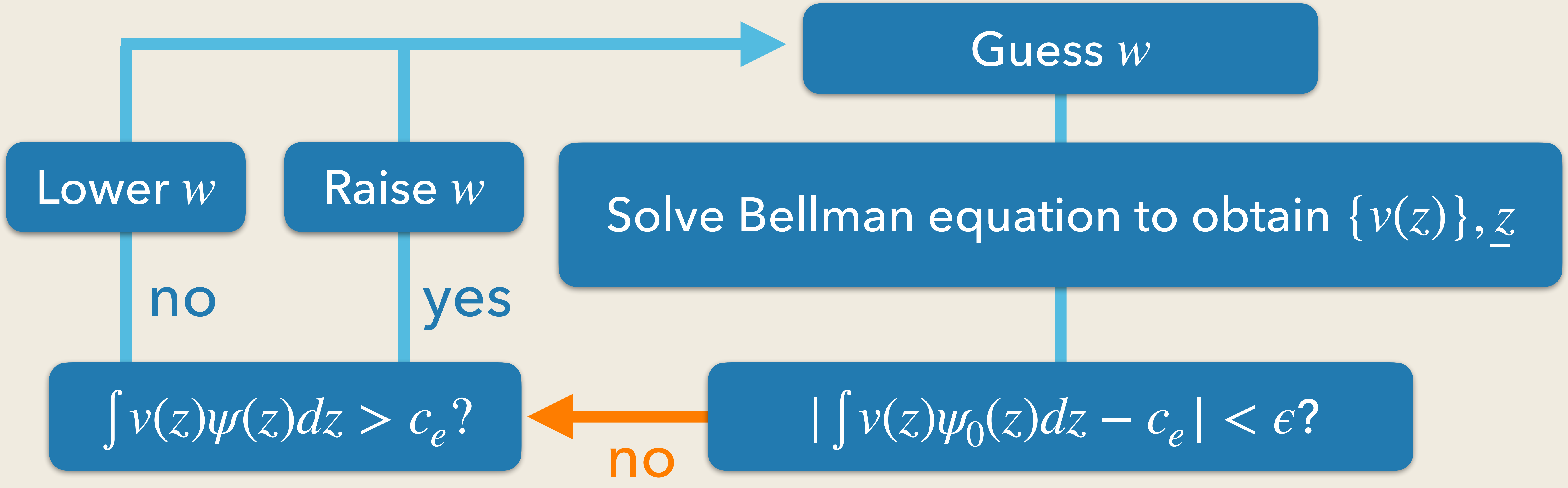
- Recover the mass of entrants m using the labor market clearing:

$$\int n(z) g(z) dz = L$$

$$\Leftrightarrow \quad m \int n(z) \hat{g}(z) dz = L$$

$$\Leftrightarrow \quad m = \frac{L}{\int n(z) \hat{g}(z) dz}$$

Computational Algorithm



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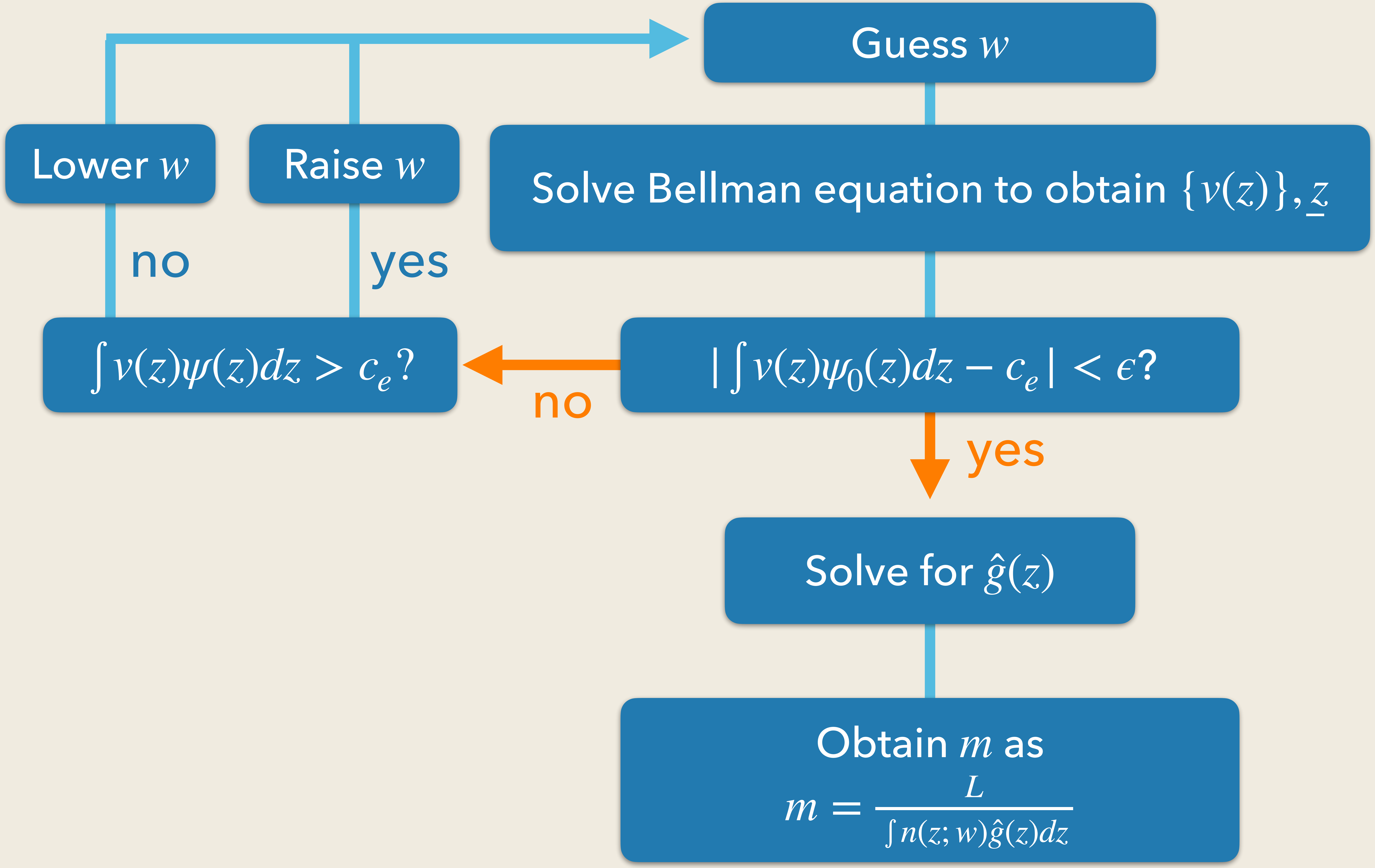


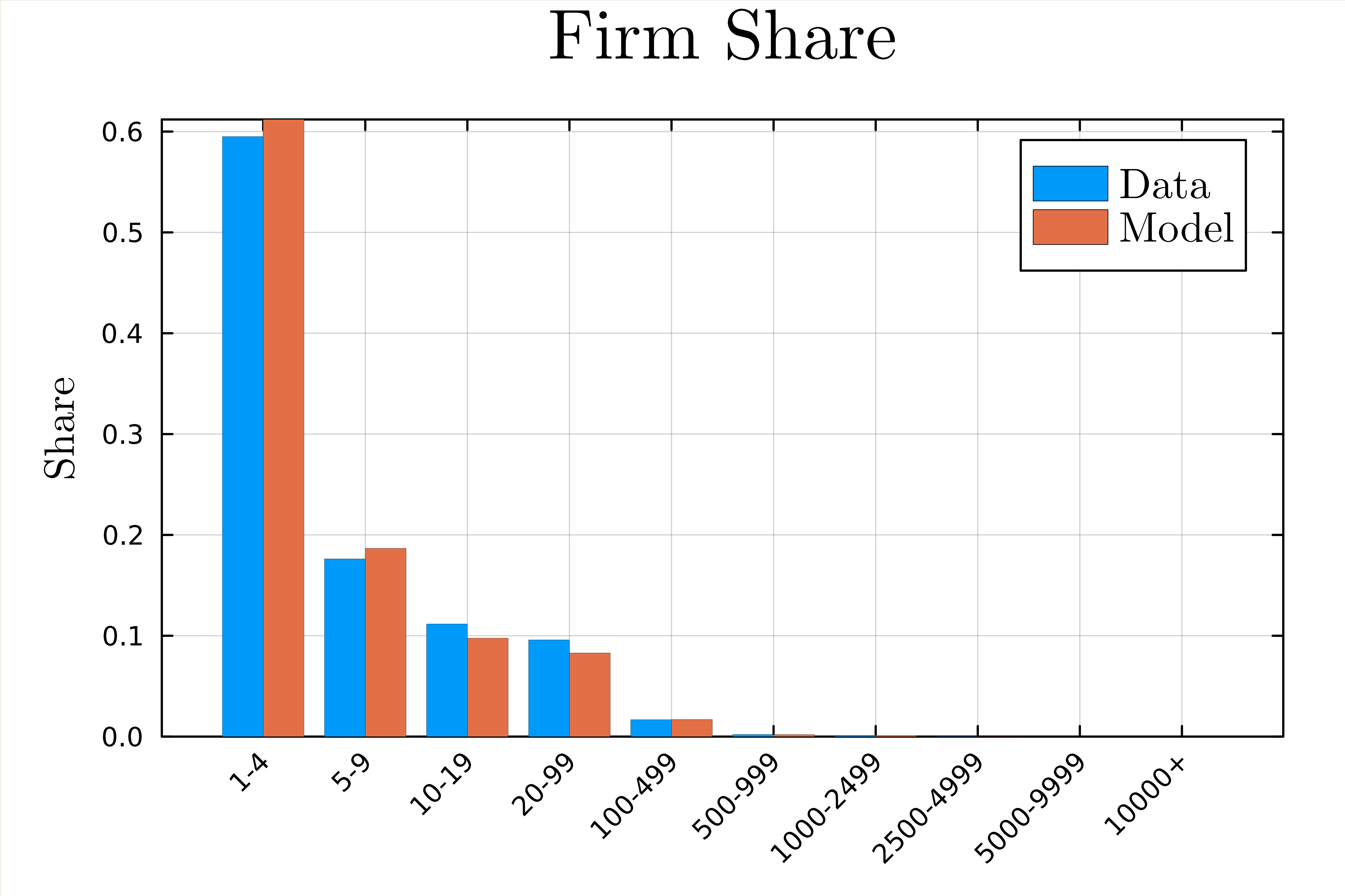
Calibration

- Assume

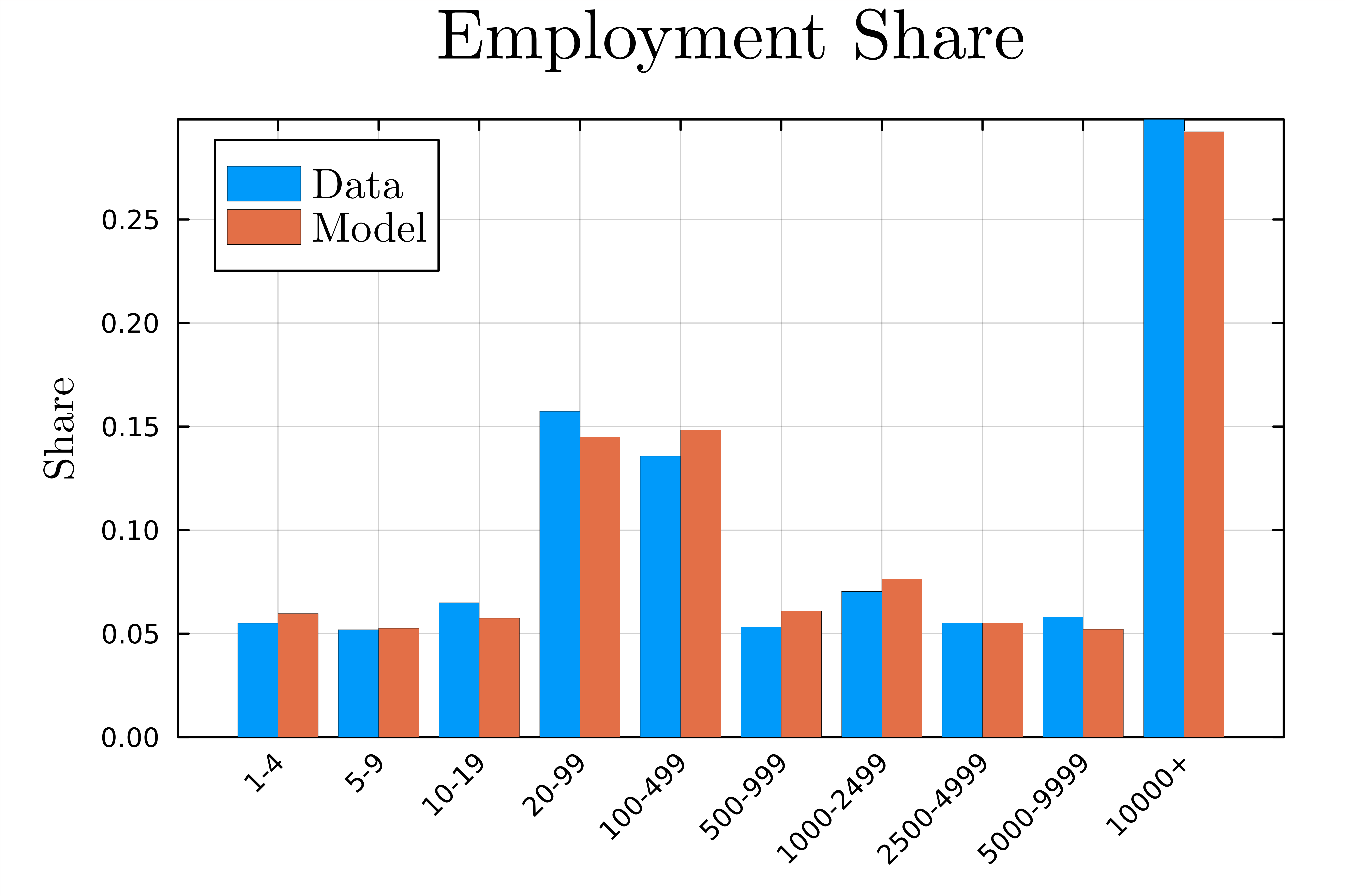
$$z_{t+1} = \gamma_{t+1} z_t, \quad \gamma_{t+1} \sim LN(\mu, \sigma^2)$$

  - Since $n \propto z$, firm growth satisfies Gibrat's law (as long as $z_{t+1} > \underline{z}$)
- Assume $\sigma = 0.41$ to match std$(\Delta \ln n) = 0.41$ reported in Elsby & Micheals (2013)
- Choose $\mu = -0.1$ to match the asymptotic Pareto tail
- Set $\beta = 0.96$ and $\alpha = 0.64$
- Normalize $L = 1$ and $c_f = 1$
- Choose c_e to match the average firm size 23 in BDS
- Assume ψ_0 is Pareto and set the Pareto tail to match the entrants size in BDS

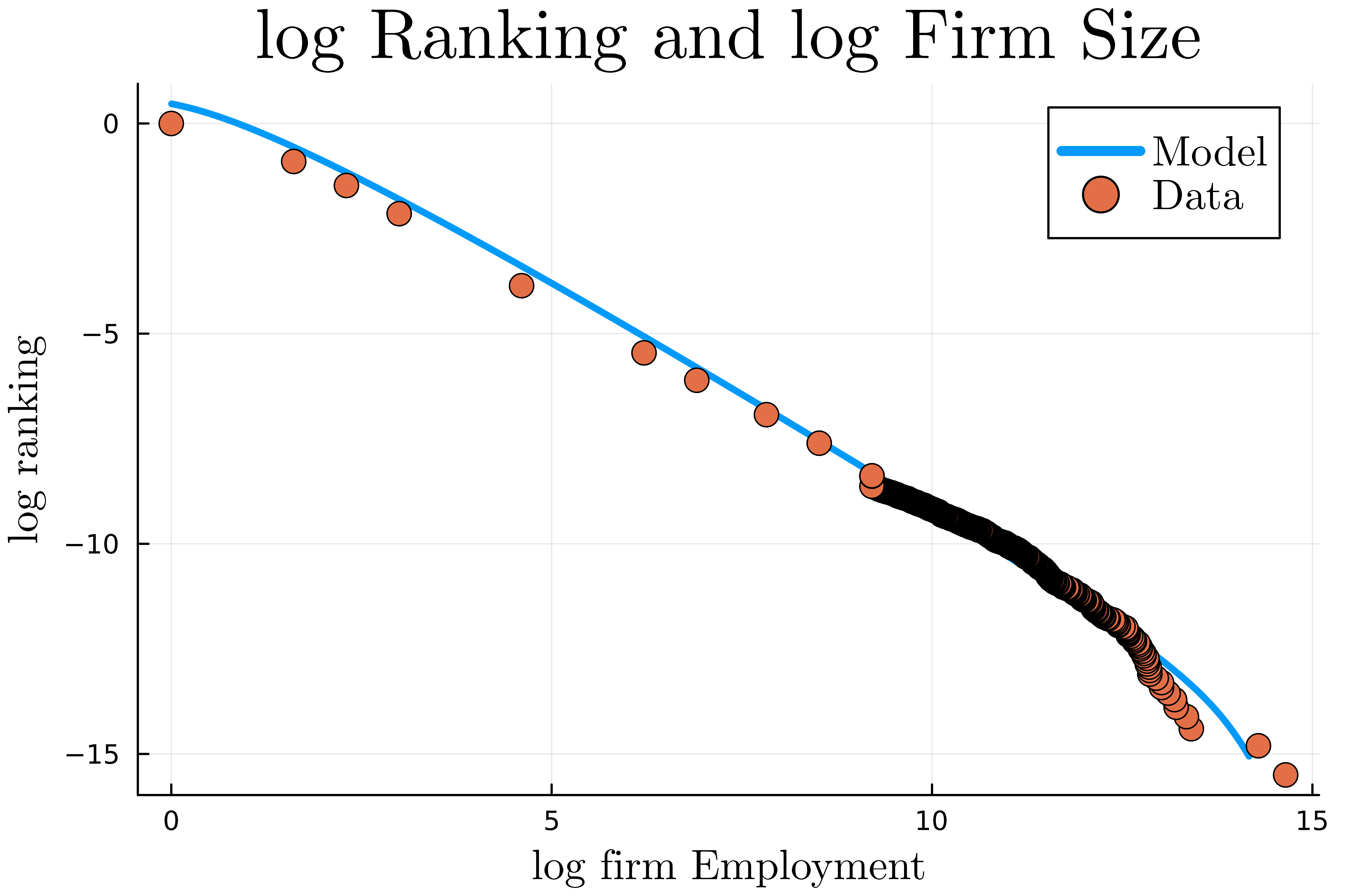
Firm Size Distribution: Data vs. Model



Employment Weighted Size Distribution



Power Law & Zipf's Law

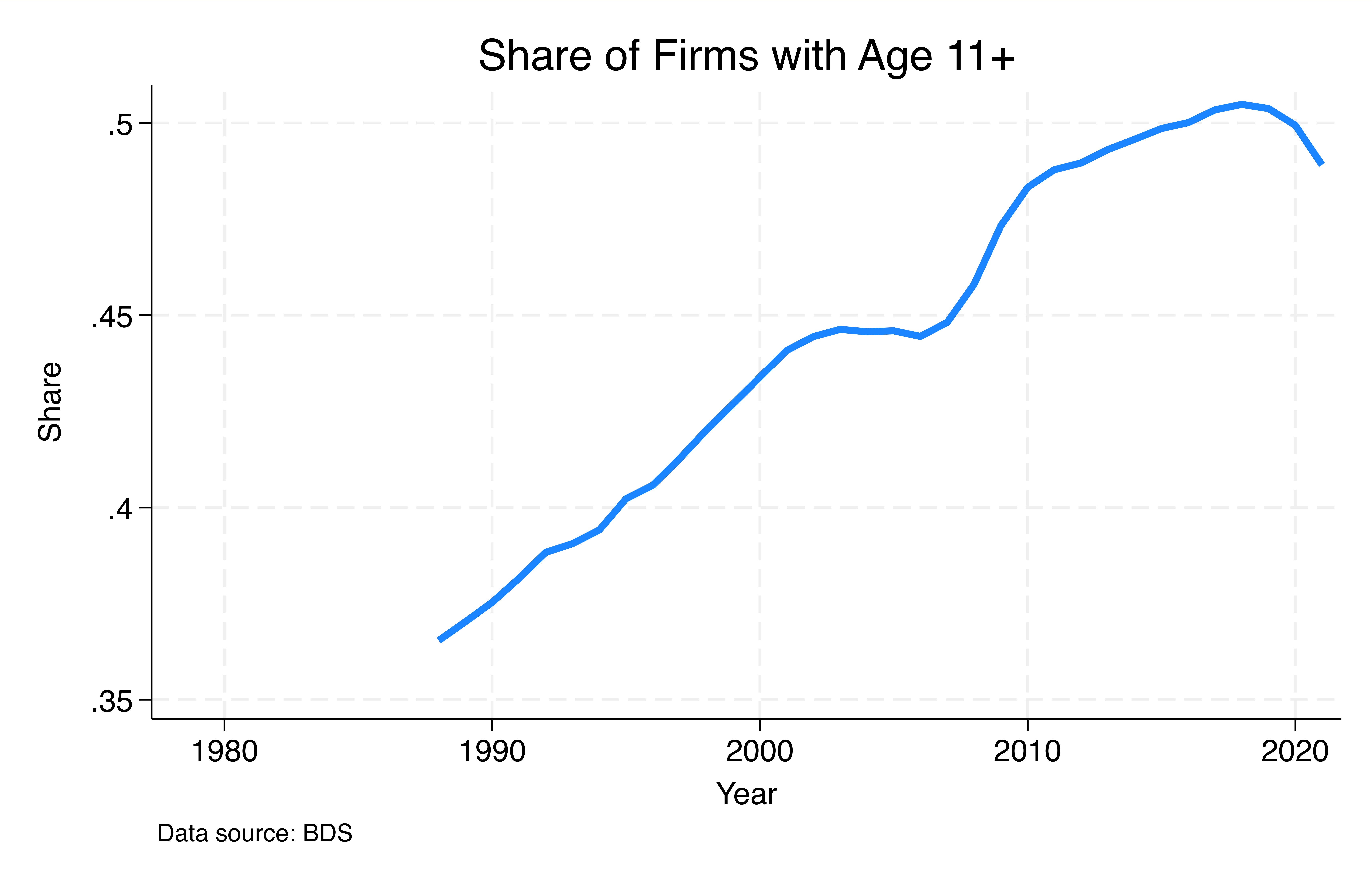


Declining Business Dynamism

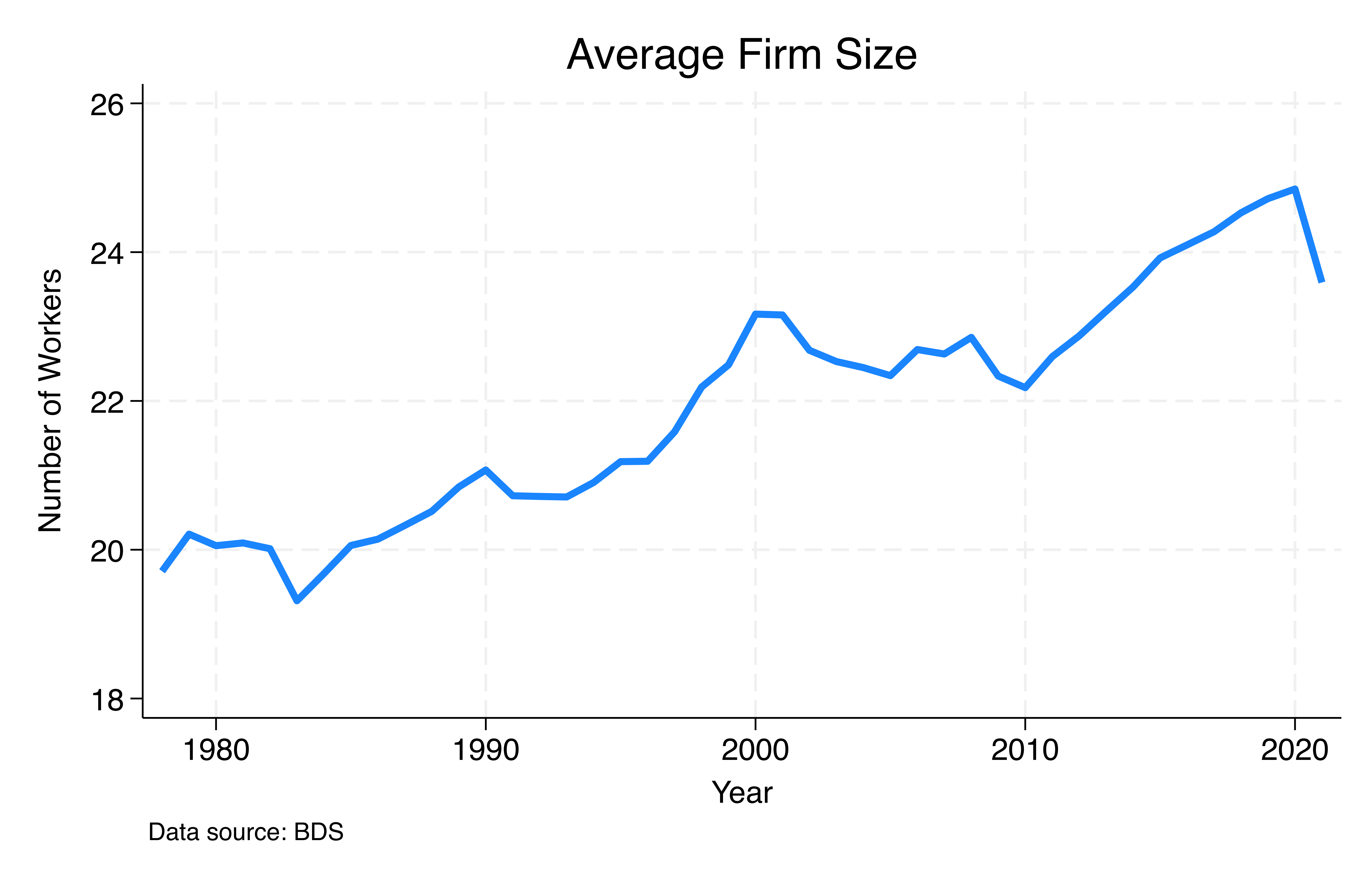
Declining Entry and Exit Rates

Data source: BDS

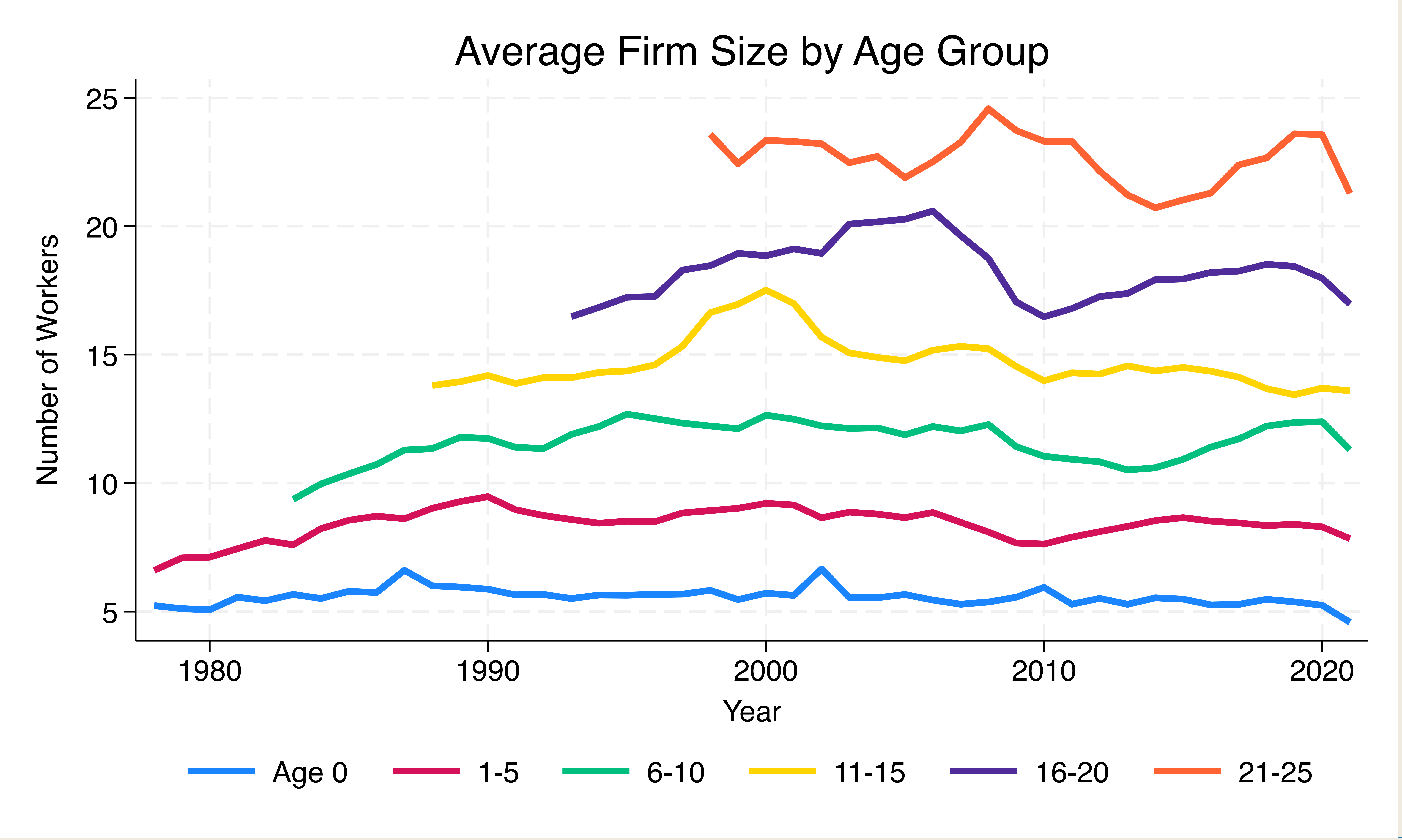
Firms are Getting Older



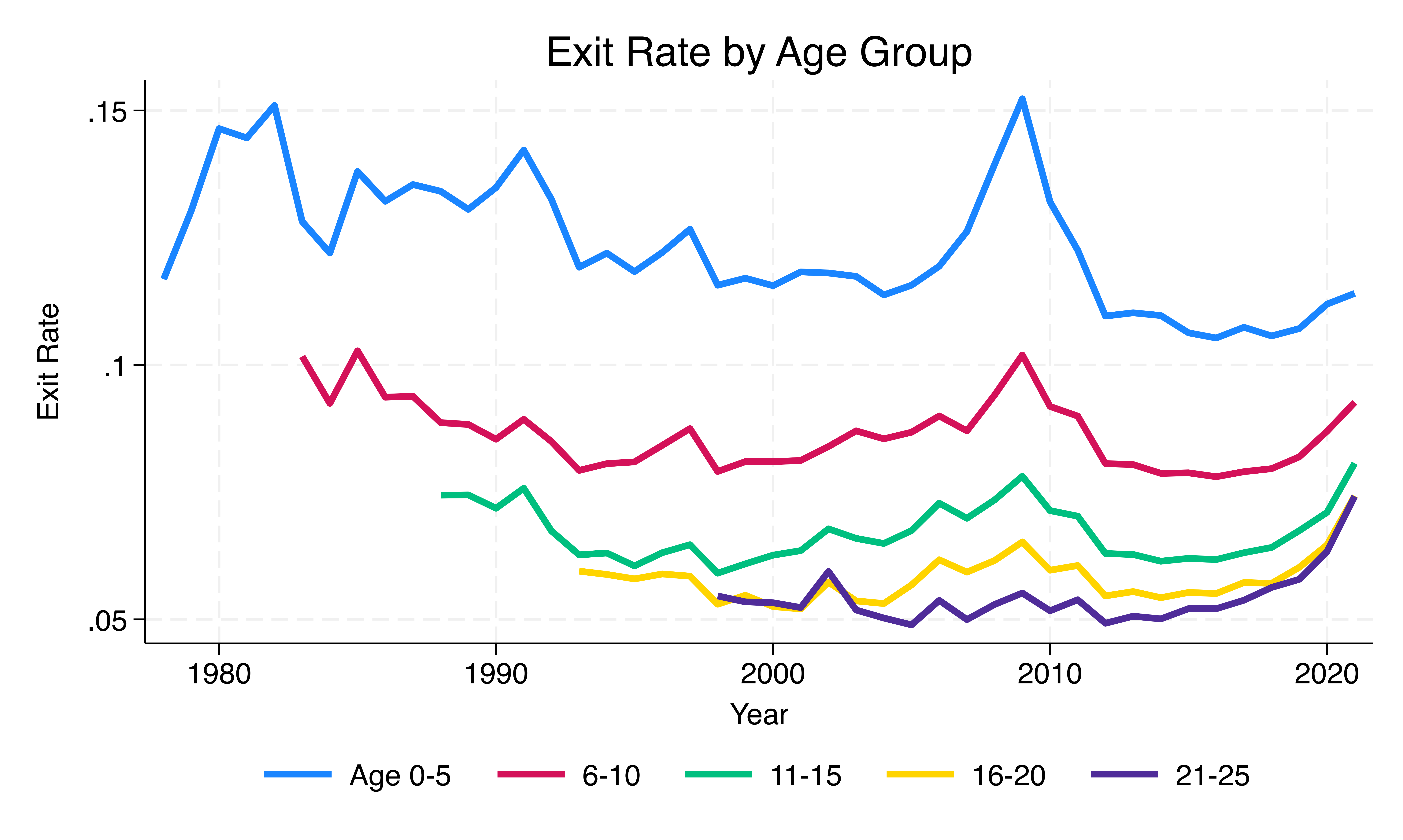
Firms are Getting Larger



Conditional on Age, Firm Size Remains Stable



Conditional on Age, Exit Rates Remain Stable

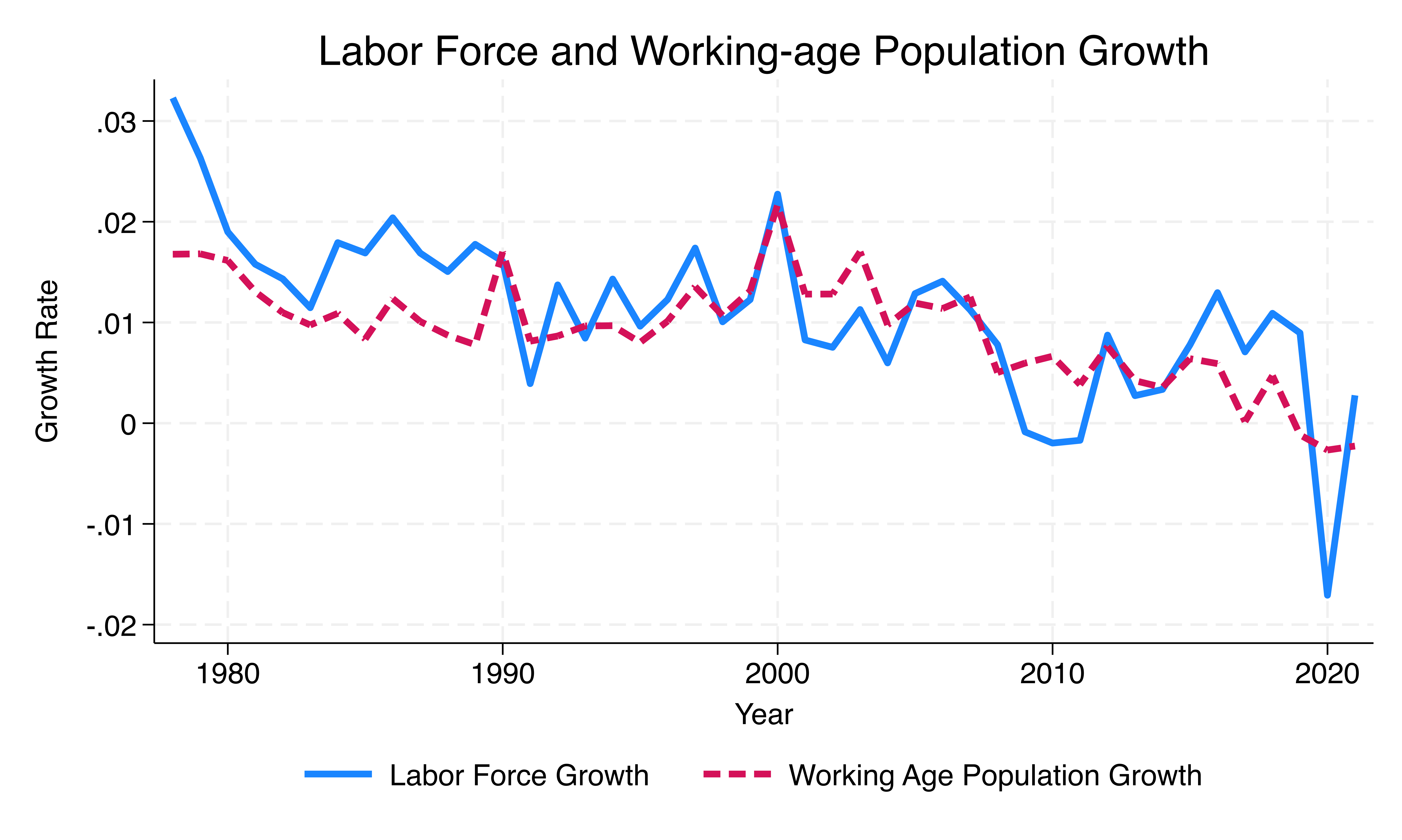


Empirical Facts

1. Entry rates have been declining, and consequently, firms are getting older
2. The firm's life-cycle dynamics (conditional on age) have little changed

Why?

Falling Labor Supply Growth



Fall in Labor Supply ⇒ Decline in Entry

- If labor supply falls, labor demand needs to fall in equilibrium

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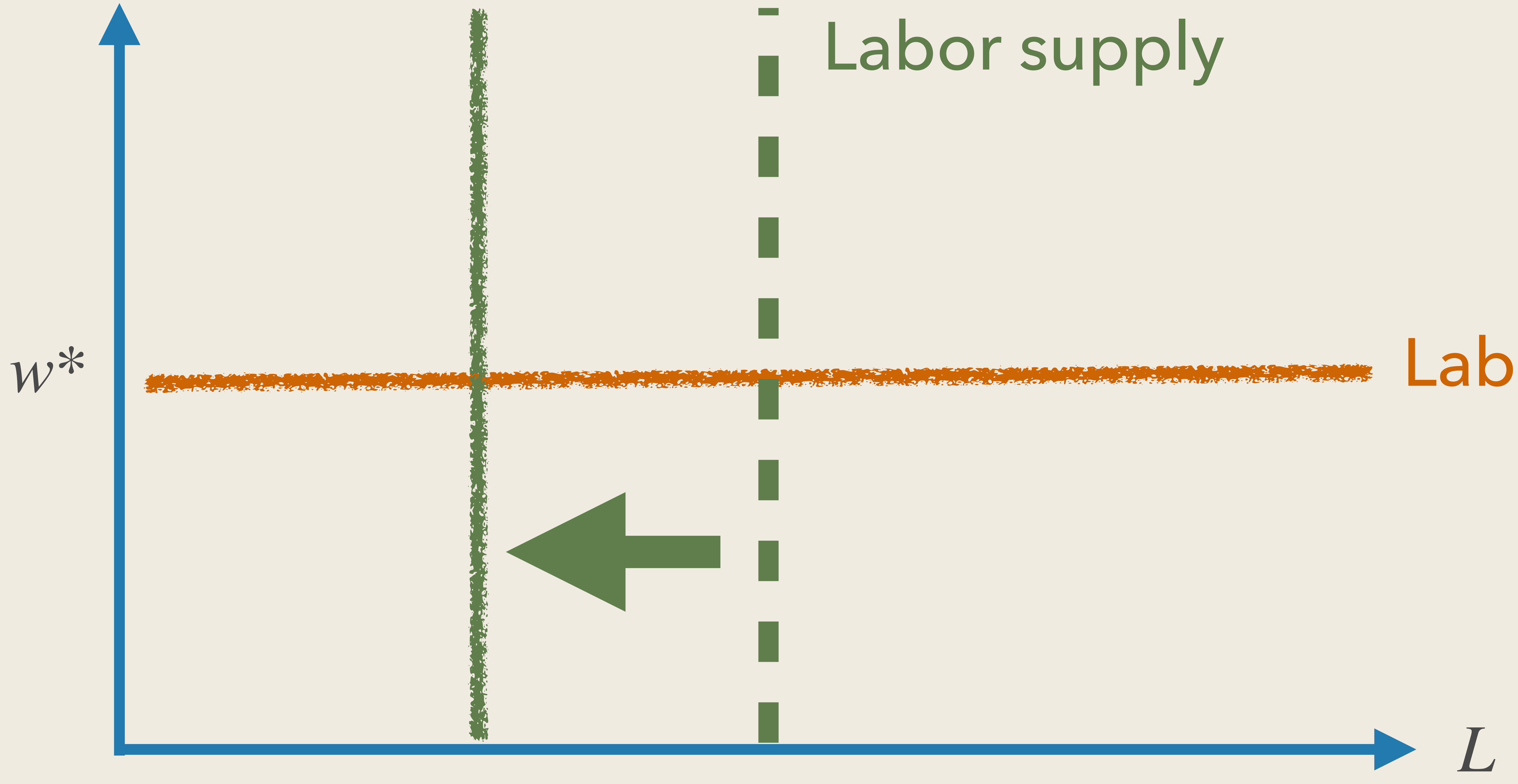

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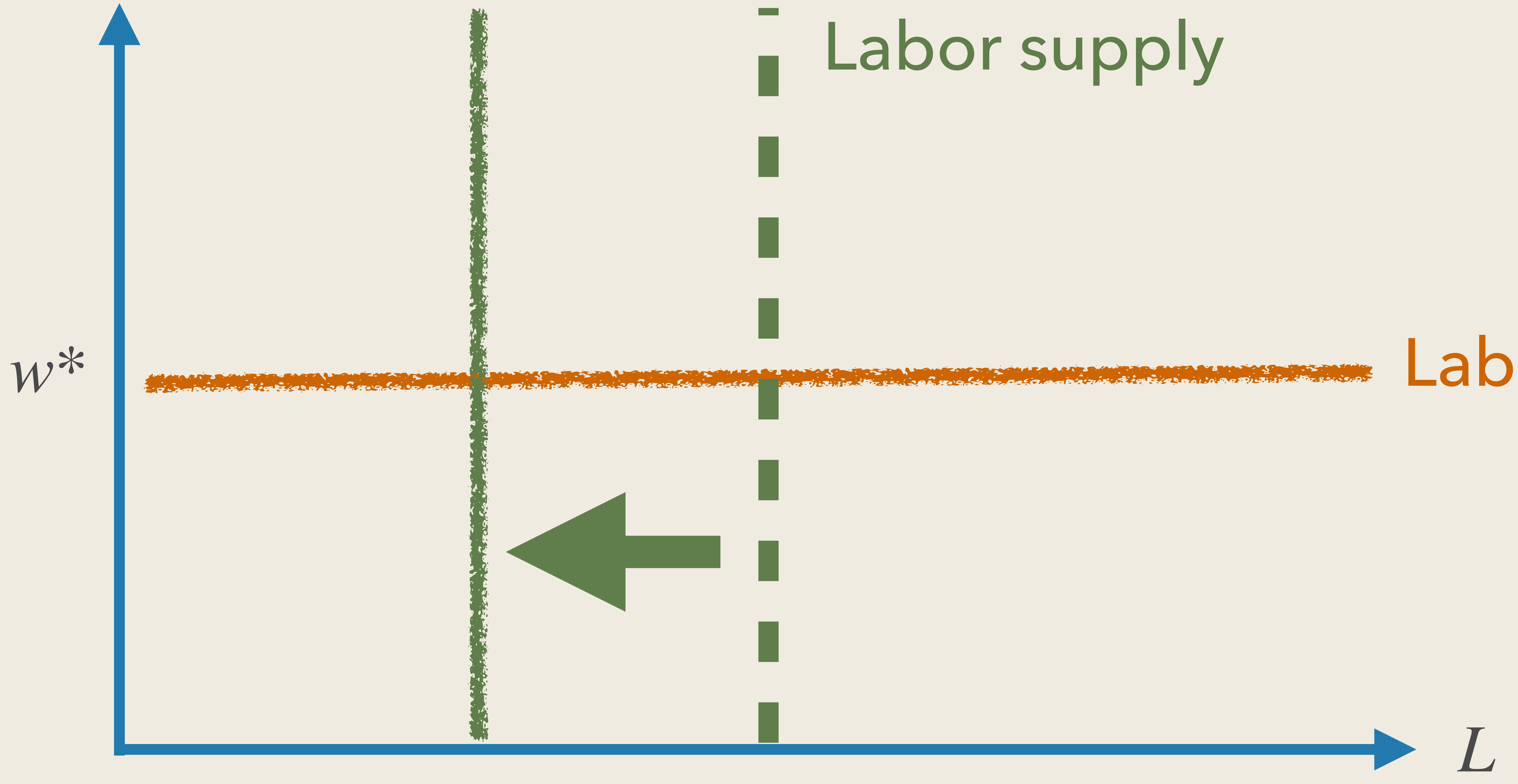

- If labor supply falls, labor demand needs to fall in equilibrium
    - In Hopenhayn-Rogerson, wages do not rise
    - Then what adjusts? – Entry

Formal Investigation

- Problem set 2 Q2 formally explores the hypothesis