Capital Accumulation and Growth: Solow Model

EC502 Macroeconomics
Topic 2

Masao Fukui

2025 Spring

Capital Accumulation as a Source of Growth

- Why do countries grow? Why are some countries richer than others?
- In the previous lectures, we saw capital plays an important role in an accounting sense
- This opens two questions
  - How do countries accumulate capital?
  - Why do some countries have higher capital stock than others?
- Idea: countries invest some of their resources into capital over time

Analogy





- Farm has a silo containing bushels of seed corn
- Farmers plant the seed, tend the crop, and harvest
- They eat 75% of the harvest and save the remaining 25% for next year's planting
- Repeat
- Each seed produces ten ears of corn, each with hundreds of kernels, so harvest grows

Solow Model

Production: $Y_t = A(K_t)^{\alpha}(L_t)^{1-\alpha}$

Capital accuulation: $K_{t+1} = (1 - \delta)K_t + I_t$

Population growth: $L_{t+1} = (1 + n)L_t$

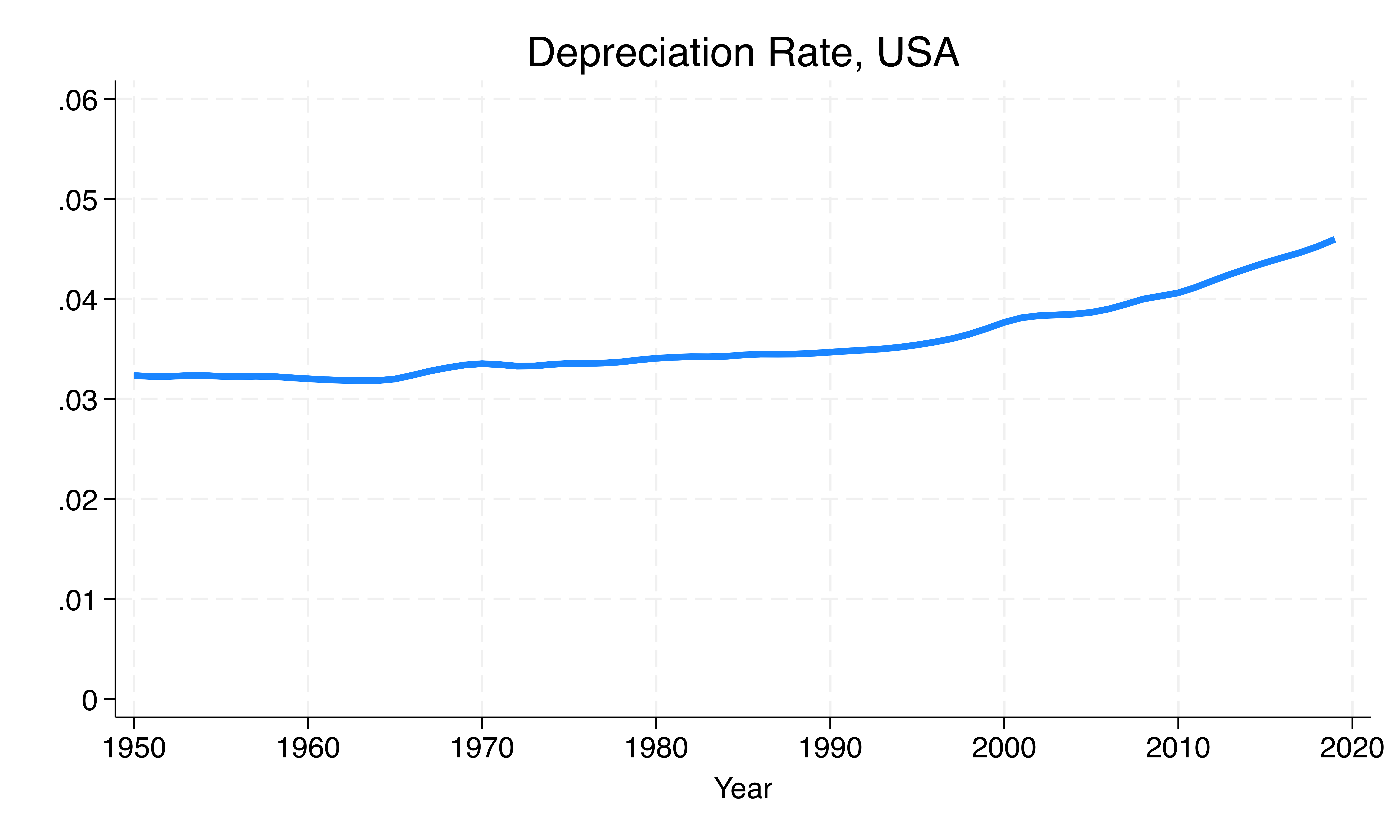
Resource constraint: $C_t + I_t = Y_t$

Investment: $I_t = sY_t$

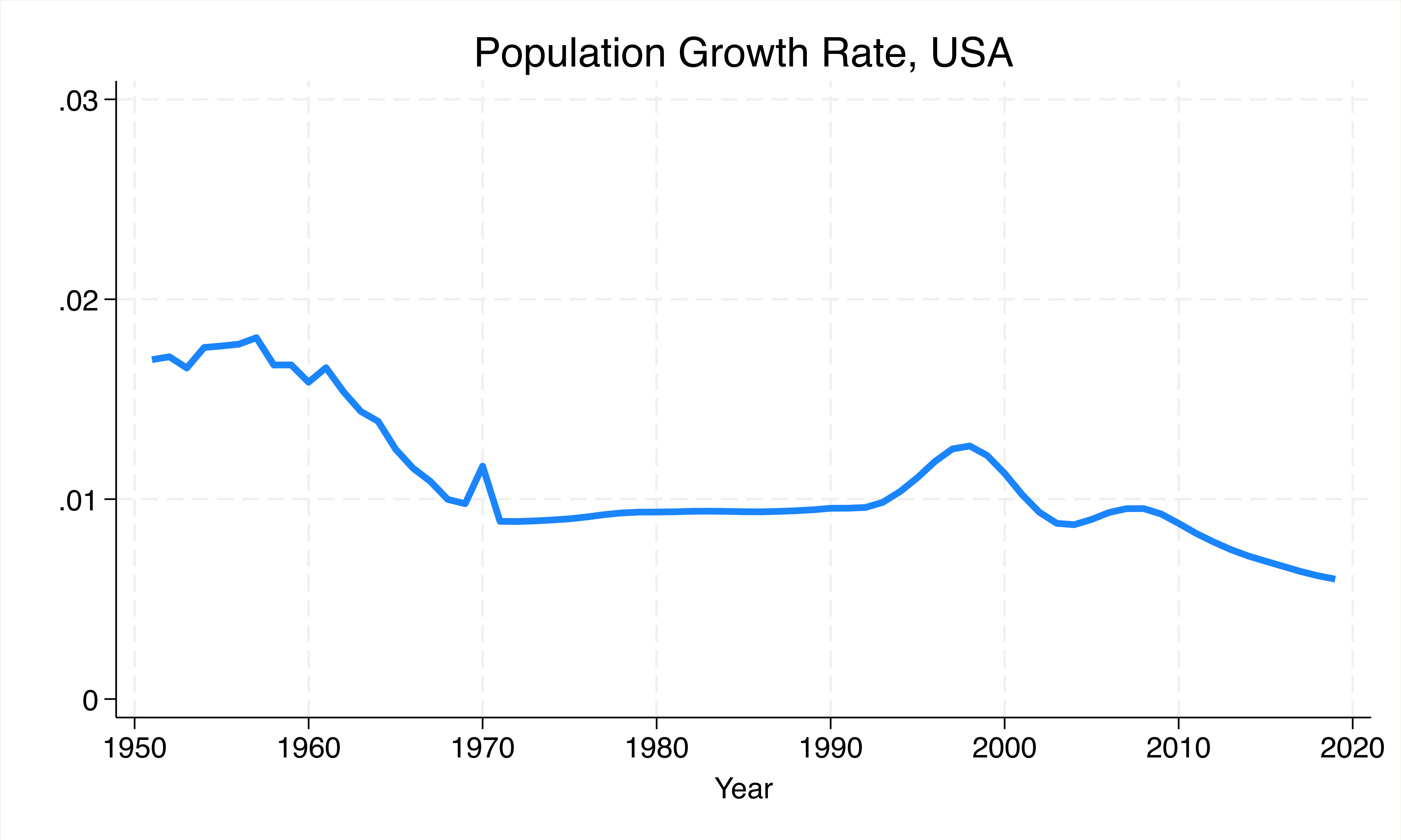
What Did We Assume?

- Production $Y_t = A(K_t)^\alpha (L_t)^{1-\alpha}$ comes from the previous lecture
- Capital accumulation $K_{t+1} = (1-\delta)K_t + I_t$ assumes constant depreciation
- We assume constant labor (population) growth $L_{t+1} = (1+n)L_t$
  - Plus, everyone in the economy supplies one unit of labor
- Resource constraint $C_t + I_t = Y_t$ is national accounting identity
  - We abstract away from G and NX
- Investment $I_t = sY_t$ assumes constant fraction of output is invested every period
- Are these assumptions reasonable?

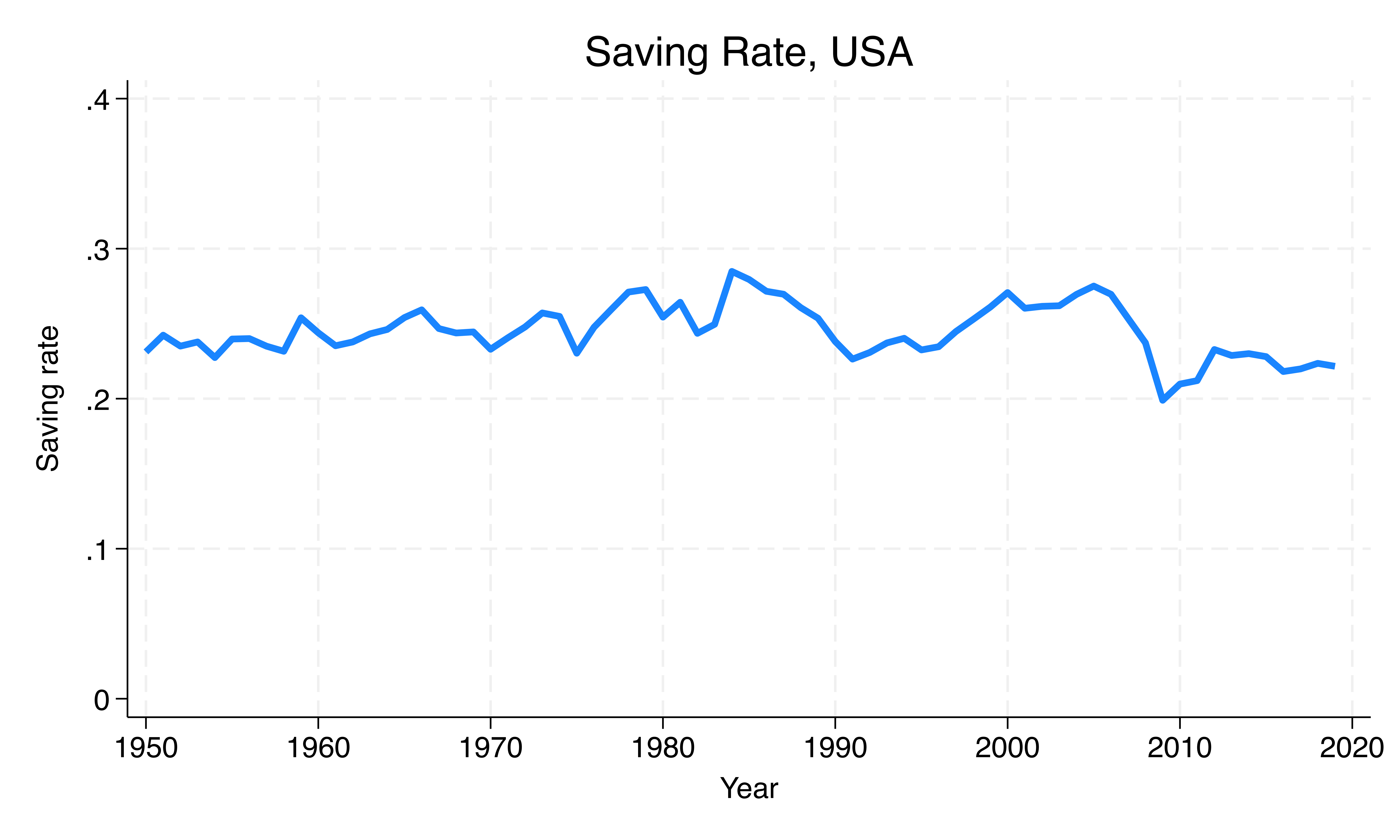
Depreciation Rate, δ



Population Growth Rate



Saving Rate, s



Normalization

- It will be convenient to divide everything by L to express in per-capita unit

$$y_t \equiv \frac{Y_t}{L_t}, \quad k_t \equiv \frac{K_t}{L_t}$$

- The production equation now becomes:

$$y_t = Ak_t^\alpha$$

- Combining capital accumulation and investment equations,

$$\underbrace{\frac{K_{t+1}}{L_{t+1}}}_{k_{t+1}} \underbrace{\frac{L_{t+1}}{L_t}}_{1+n} = k_t(1-\delta) + sy_t$$

Key Equation

- Putting the previous two equations together,

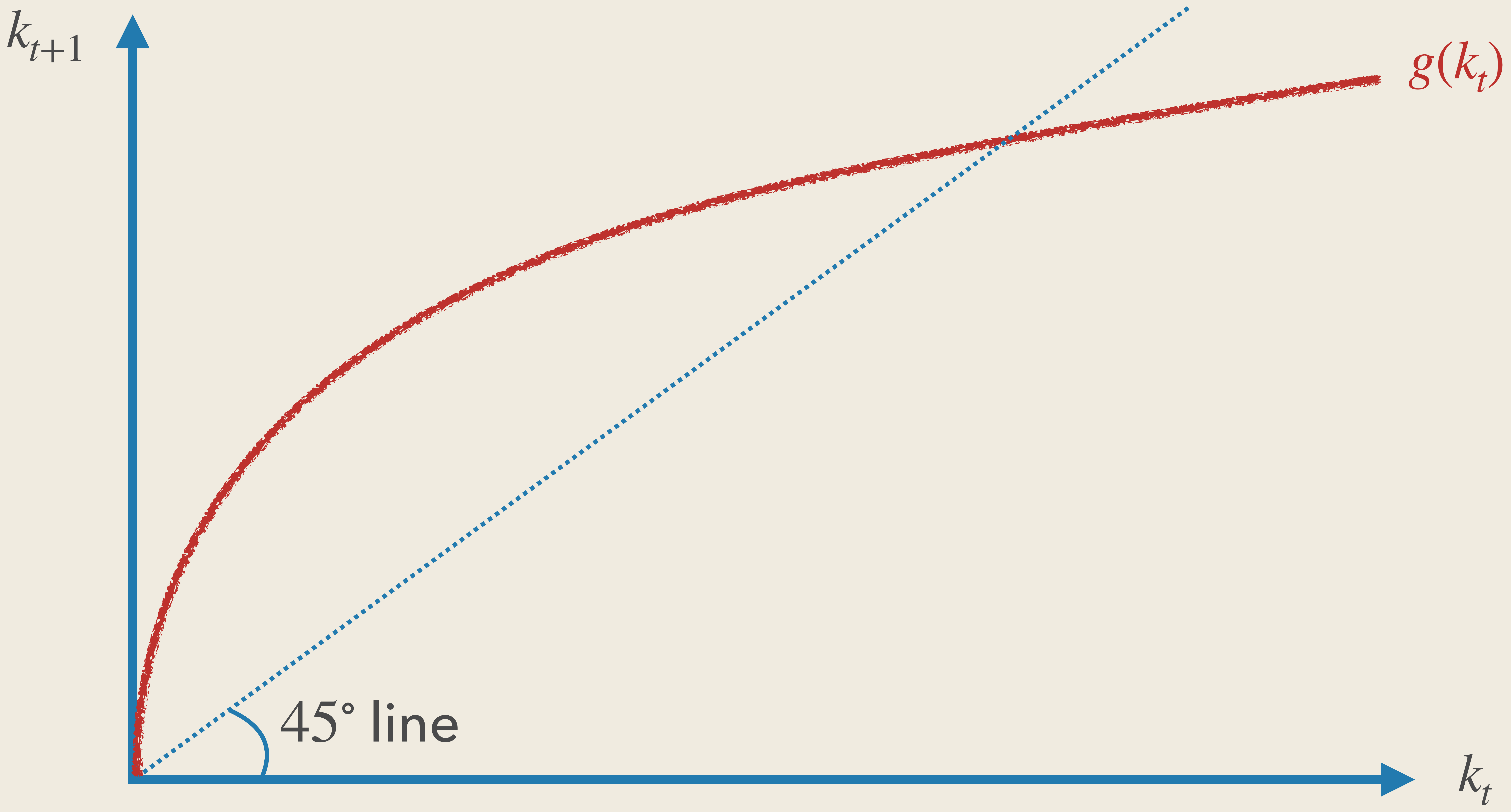
$$k_{t+1} = \frac{1}{1+n}\left[(1-\delta)k_t + sAk_t^{\alpha}\right]$$
$$\equiv g(k_t)$$

- Given k_0, the above equation determines the path of $k_1, k_2, k_3, \ldots$

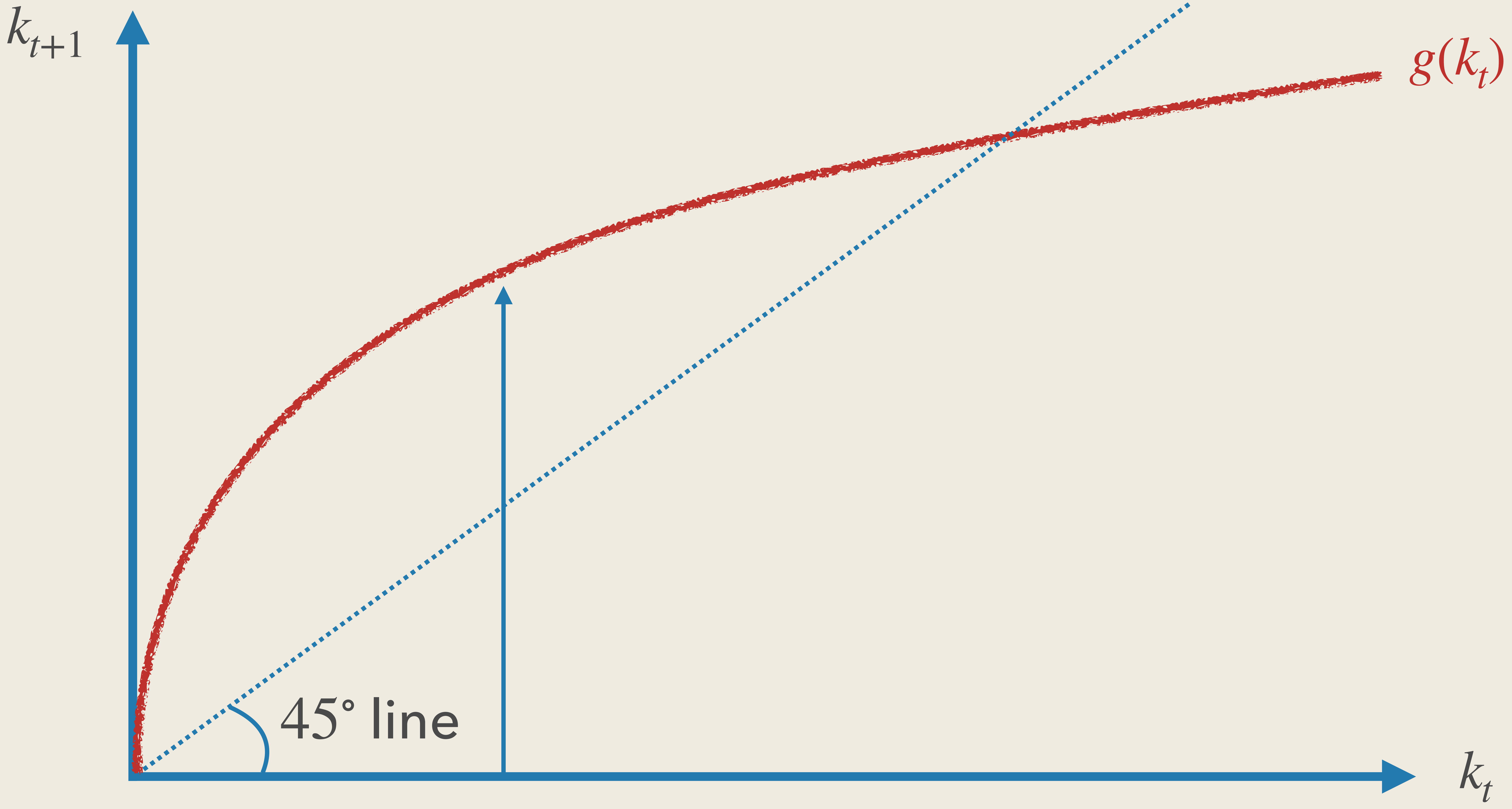
- What is the property of $g(k_t)$?
  - Increasing: $g'(k_t) = \frac{1}{1+n}\left[1-\delta+\alpha Ak^{\alpha-1}\right] > 0$
  - Concave: $g''(k_t) = \frac{1}{1+n}\alpha(\alpha-1)k_t^{\alpha-2} < 0$
  - Also satisfies

$$g(0)=0, \quad g'(0)=\infty, \quad \lim_{k\to\infty} g'(k) = \frac{1-\delta}{1+n} < 1$$

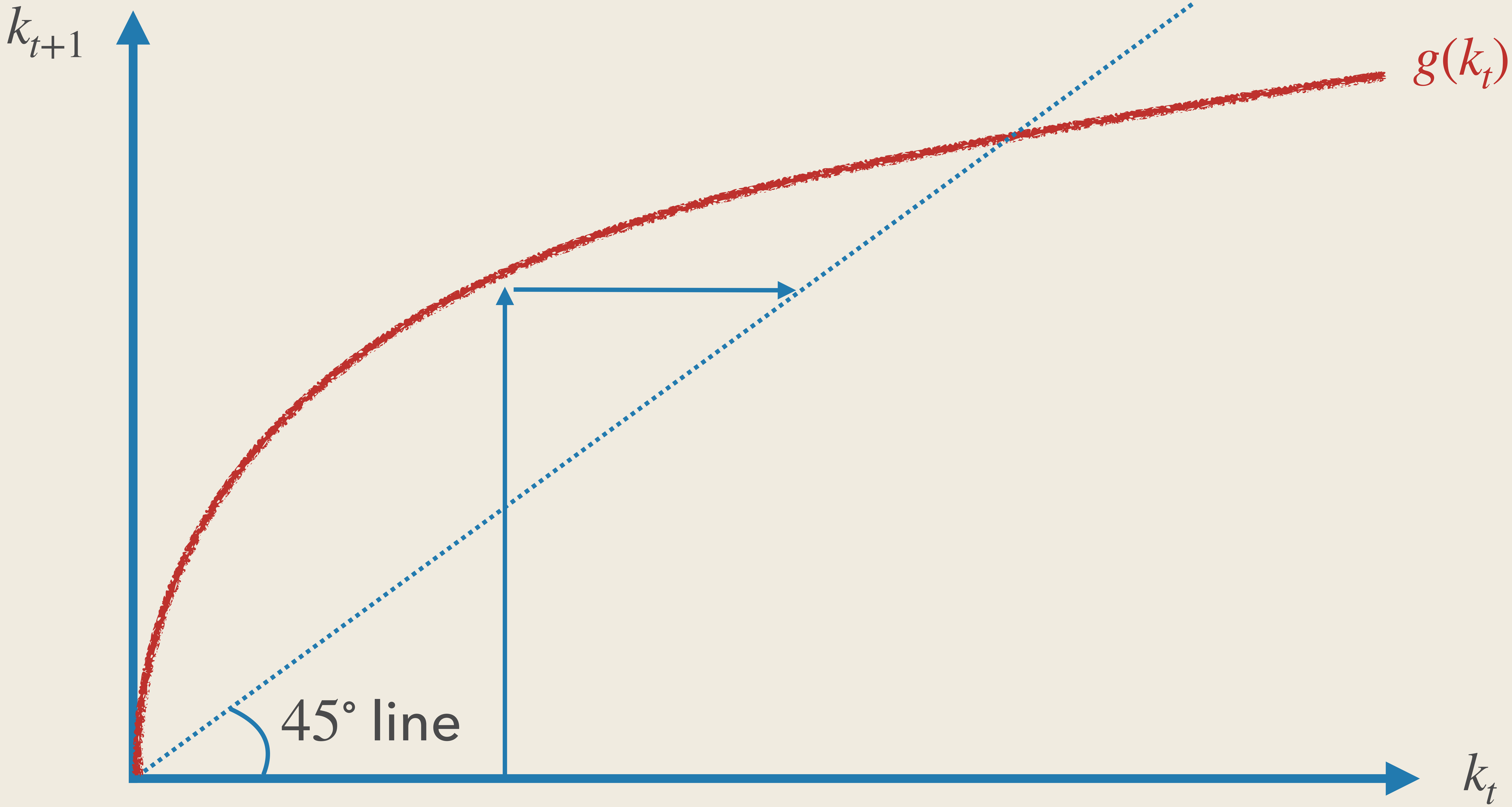
Evolution of Capital Stock



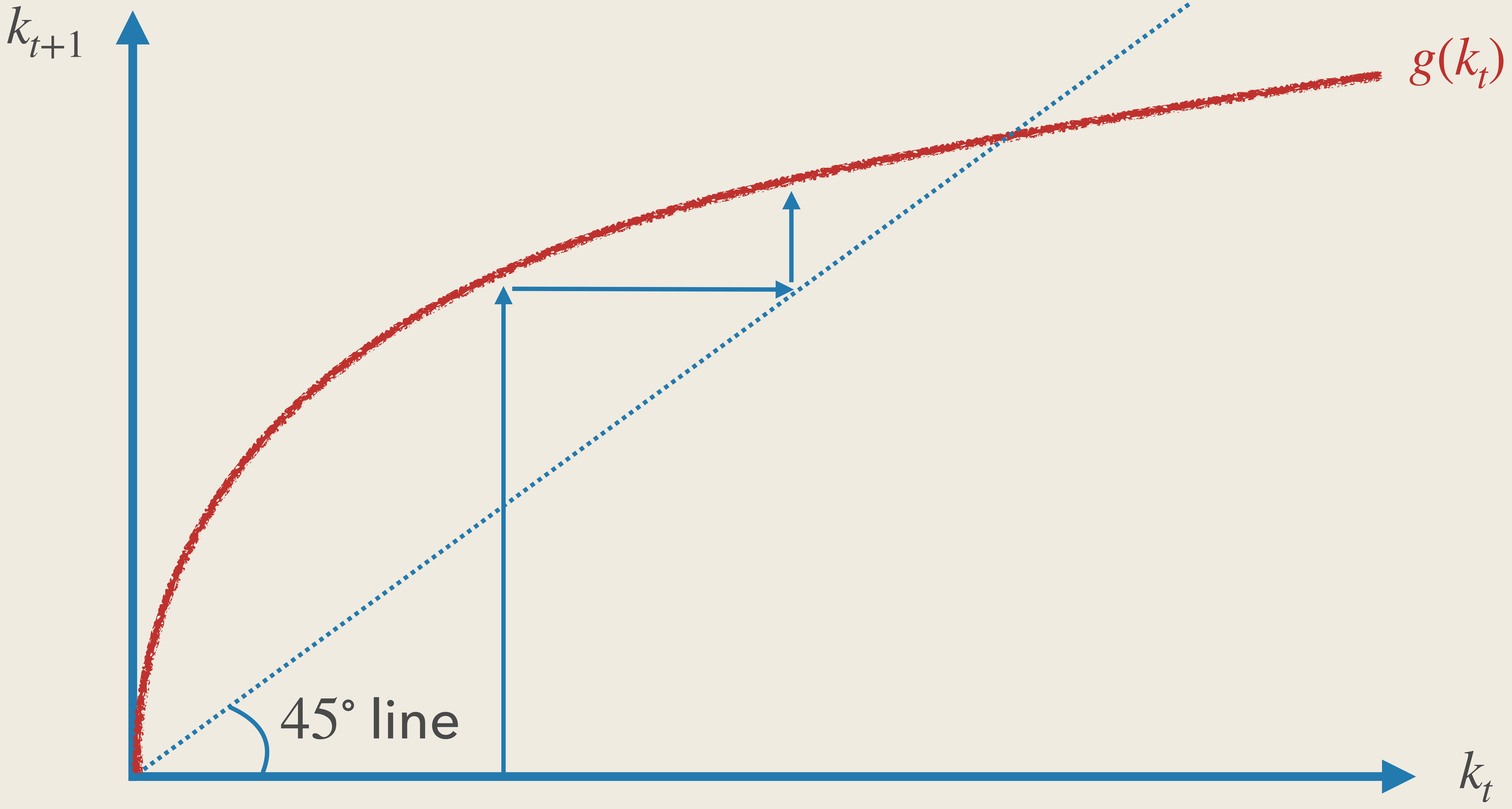
Evolution of Capital Stock



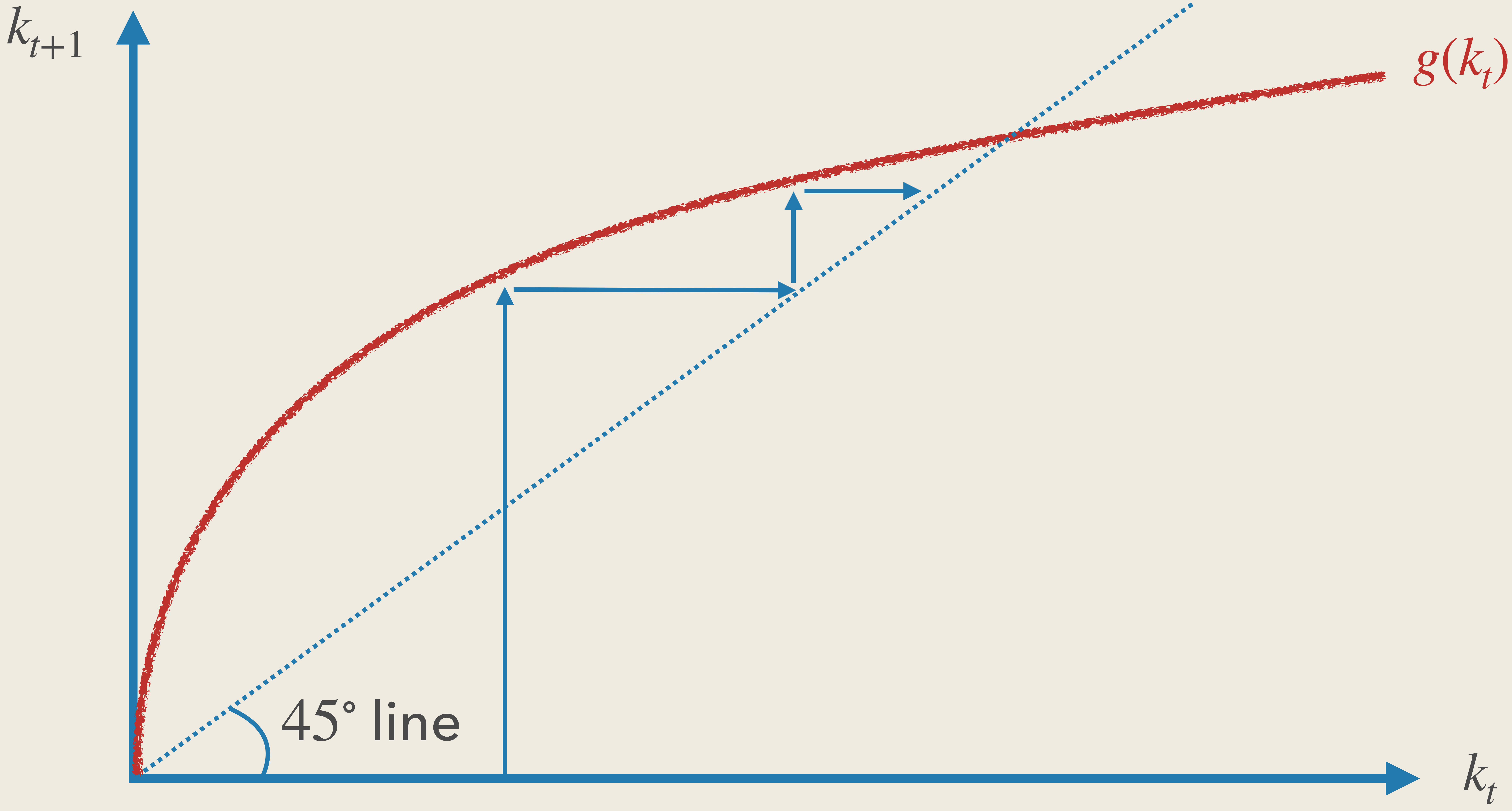
Evolution of Capital Stock



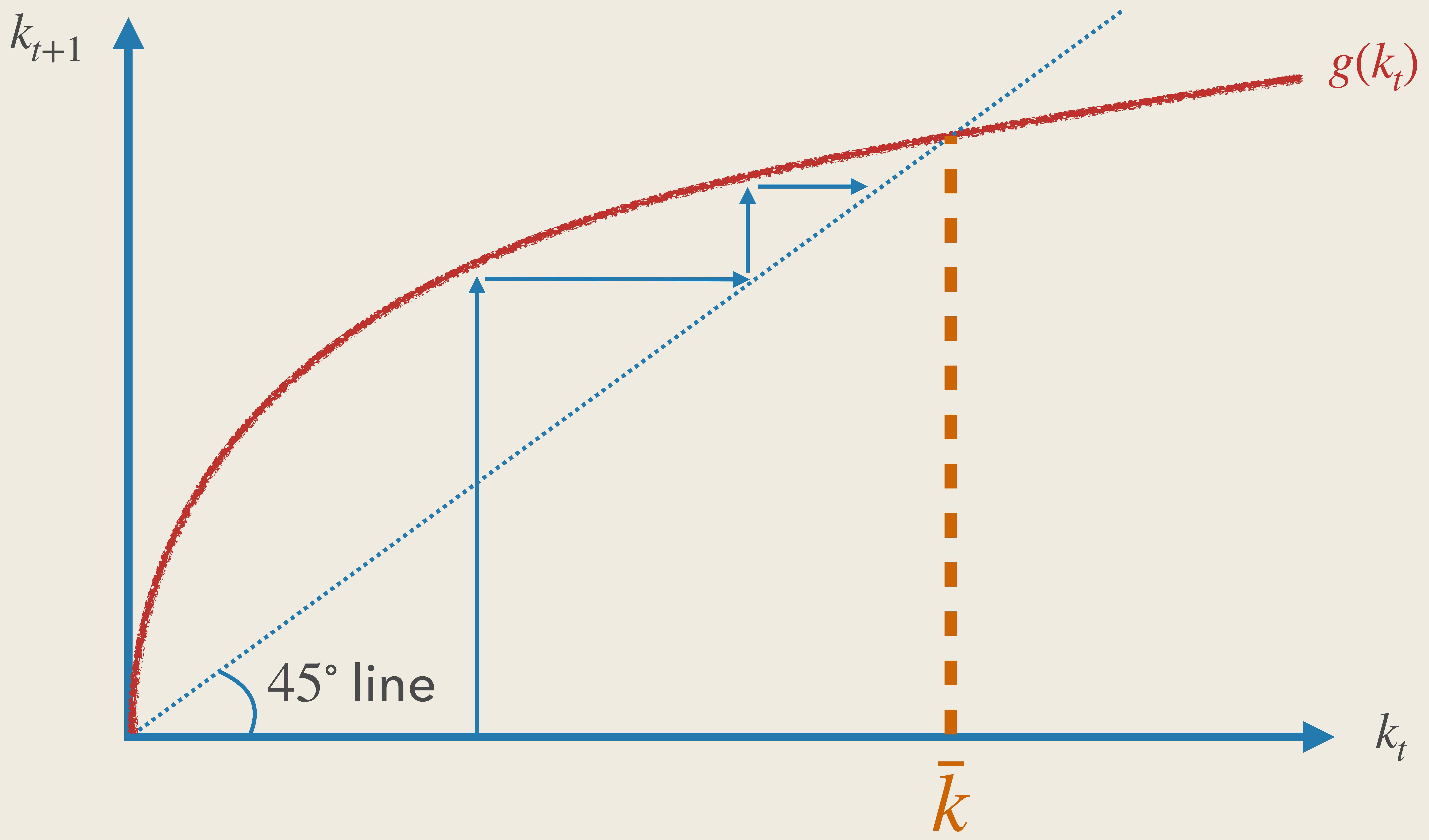
Evolution of Capital Stock



Evolution of Capital Stock



Evolution of Capital Stock



Steady State

- In the long-run (**steady state**), the capital stock converges to $\bar{k}$ that satisfies

$$\bar{k} = \frac{1}{1+n}\left[(1-\delta)\bar{k} + s\,\underbrace{A\bar{k}^{\alpha}}_{\bar{y}}\right]$$

- Dividing both sides by y and rearranging, we get

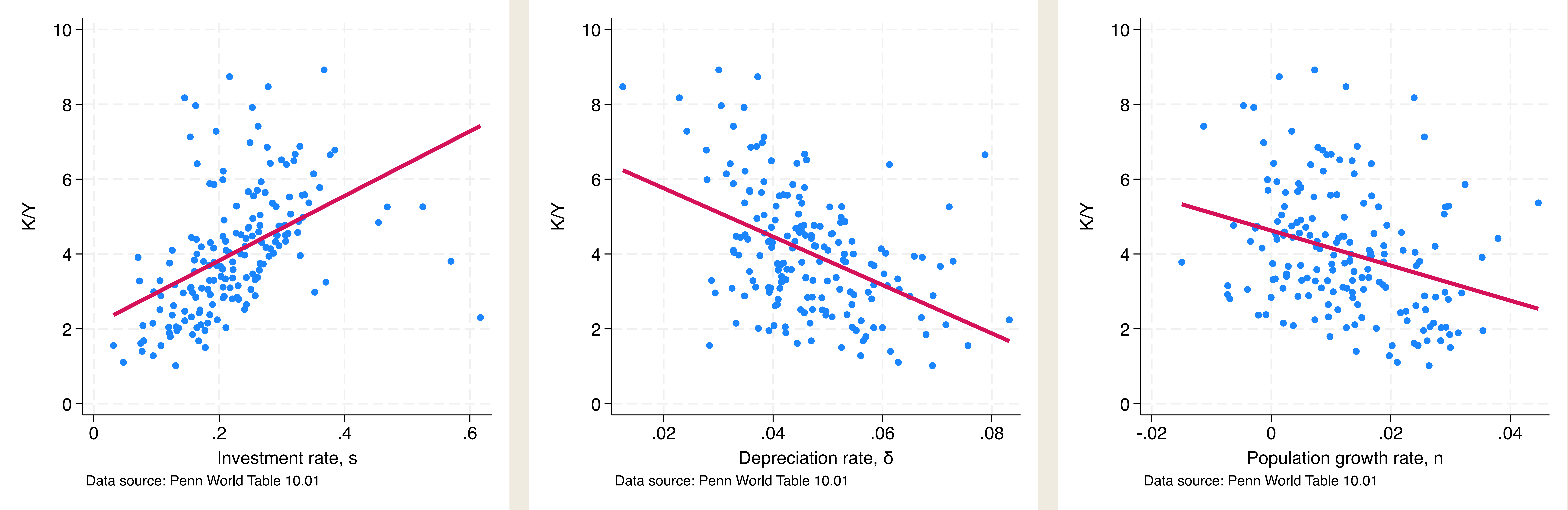
$$\frac{\bar{k}}{\bar{y}} = \frac{s}{n+\delta} \qquad \text{or} \qquad \bar{k} = \left(\frac{As}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

  Long-run capital-to-GDP ratio (capital intensity) is high if

  - investment rate (s) is high
  - depreciation rate (δ) is low
  - population growth (n) is low

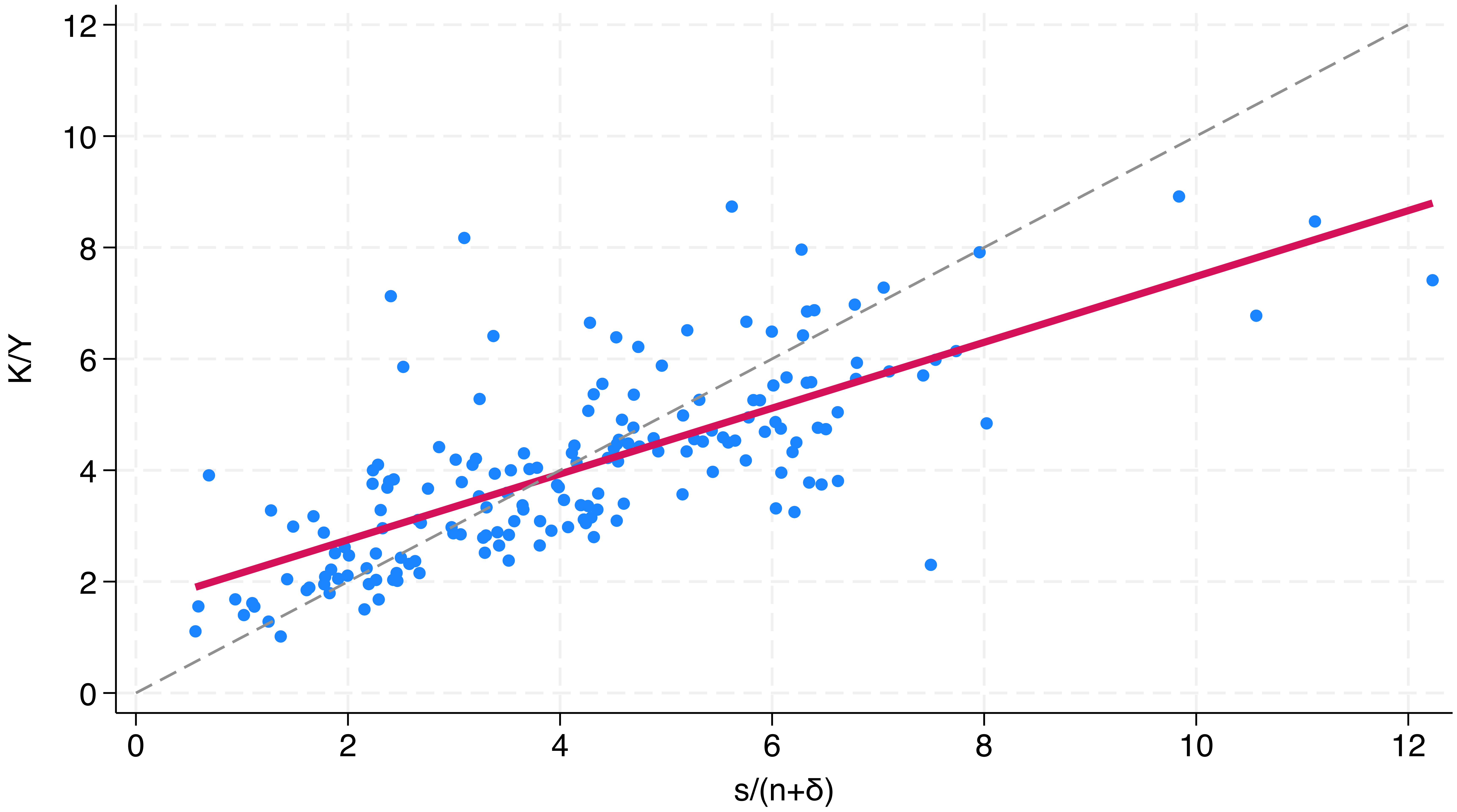
Testing Solow Model

K/Y and s | K/Y and δ | K/Y and n



- Assuming all countries are in steady-states in 2019, we confront the model with data

K/Y in the Model and in the Data



Data source: Penn World Table 10.01

Economic Growth in Solow Model

Long-Run Growth in Solow Model

- What is the long-run growth rate of the economy according to the Solow model?

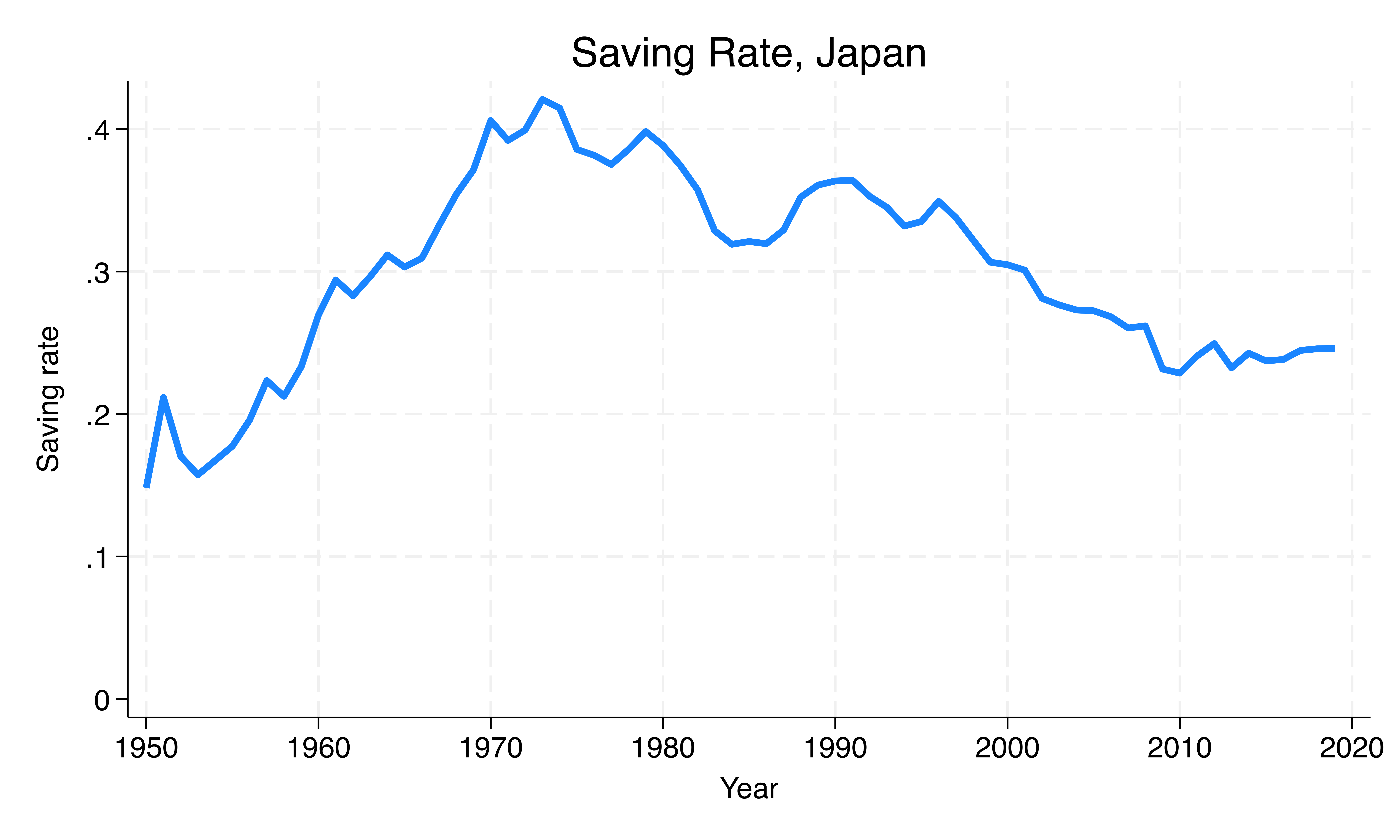
Zero! There is no long-run growth in Solow!

- Capital stock per capita, k, is constant in the steady state, and so is output, $y = Ak^{\alpha}$
- This is because of decreasing returns to scale
  - As we accumulate more and more k, y rises by a smaller and smaller amount
  - But capital depreciate at a constant rate
- Diminishing returns to capital is at the heart of why growth eventually ceases
- A huge, disappointing failure.

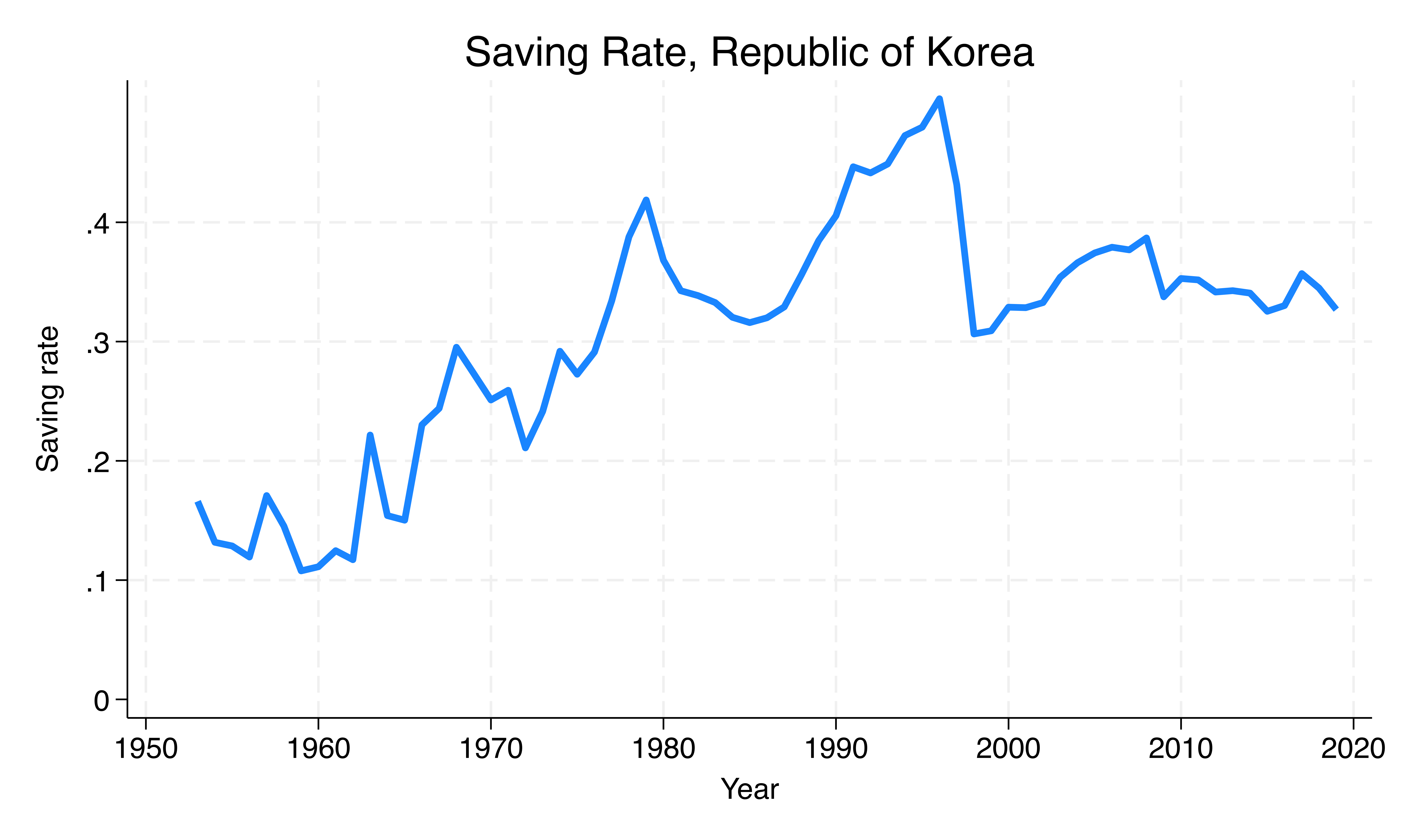
Transition Dynamics

- Despite this negative result on long-run growth, the Solow framework is useful
- Solow model does predict growth along the transition dynamics
- Suppose a country begins in a steady state
- What happens if this country suddenly starts to invest more (a rise in s)?
- This has happened in many East Asian growth miracle countries

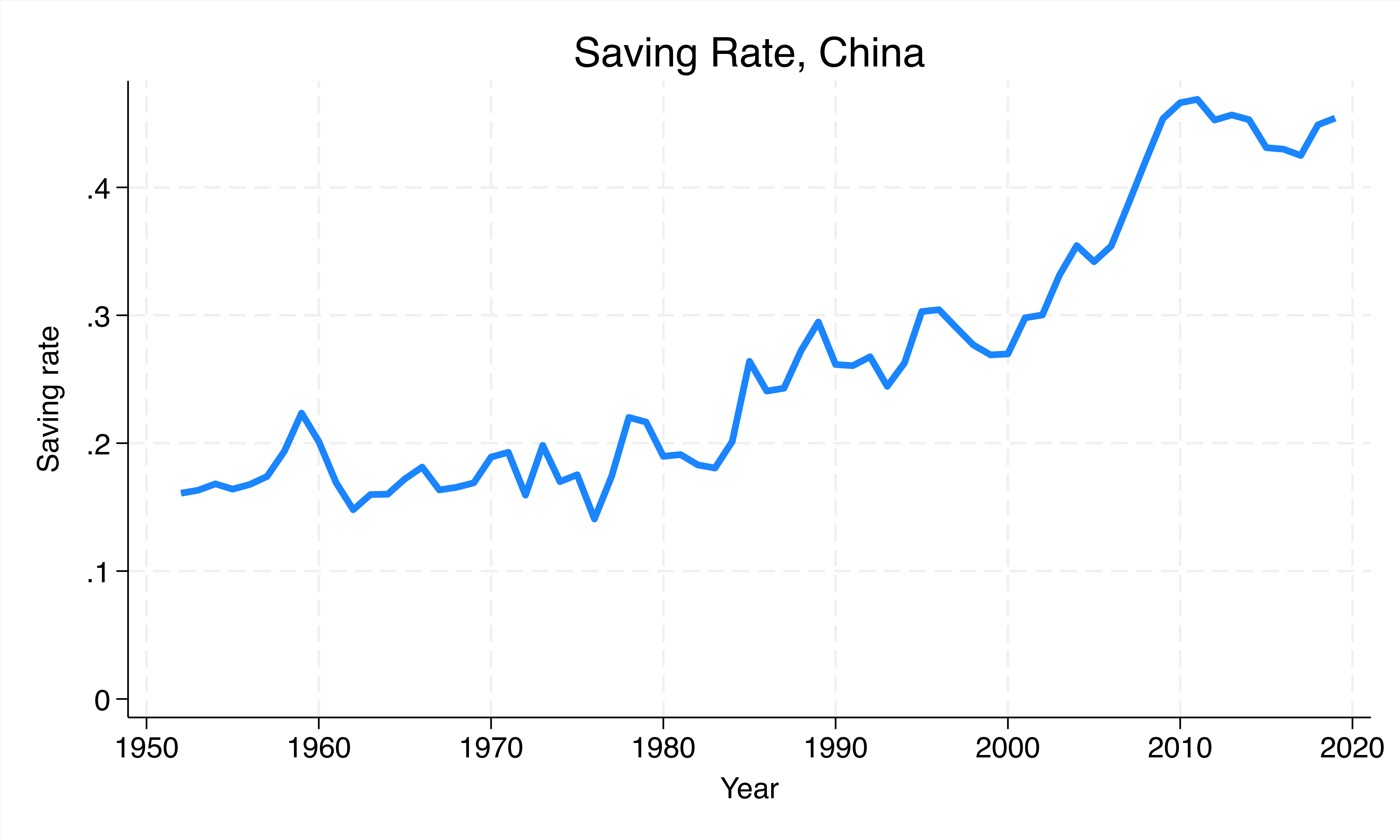
Saving Rate: Japan



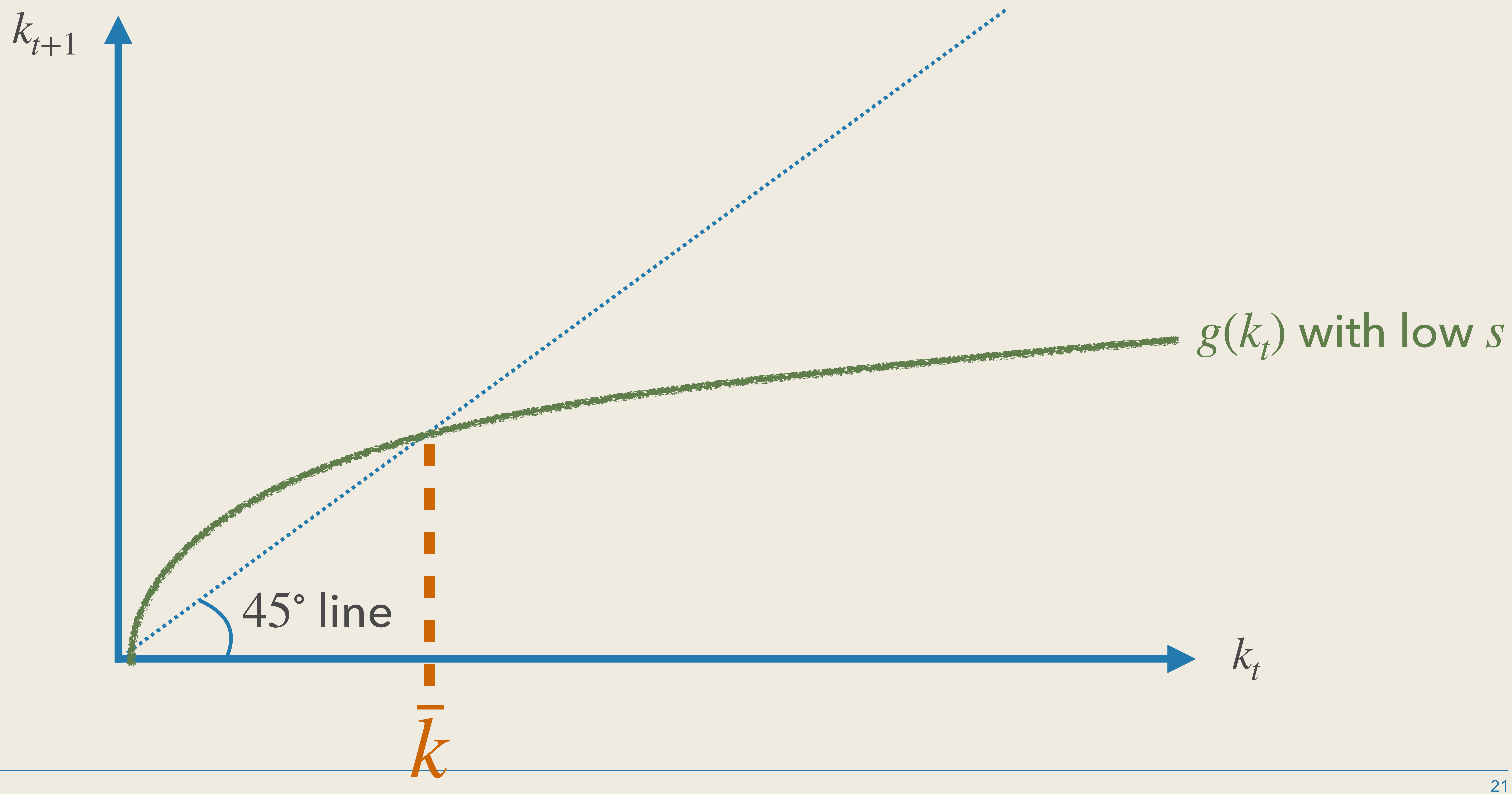
Saving Rate: South Korea



Saving Rate: China



Evolution of Capital Stock



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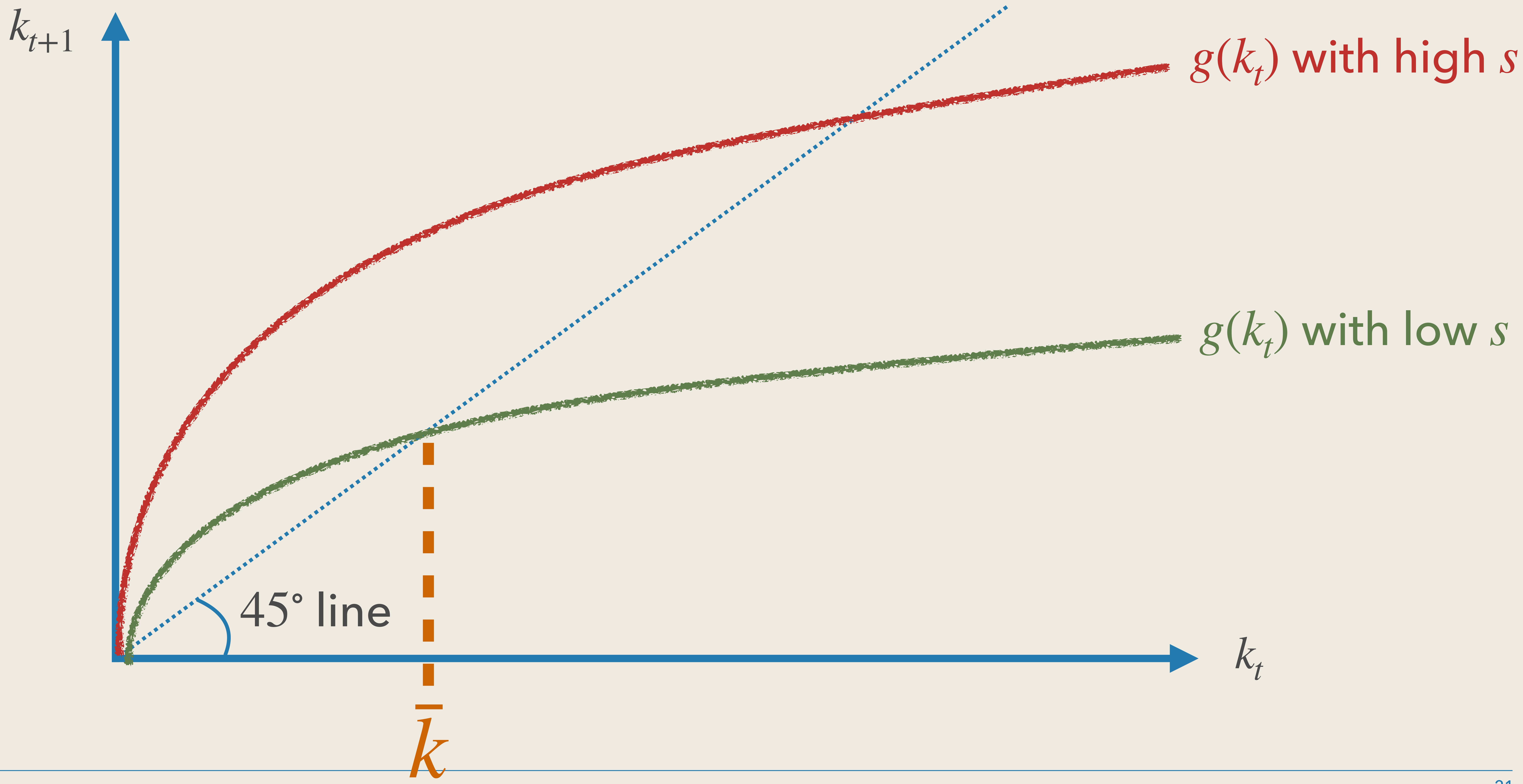

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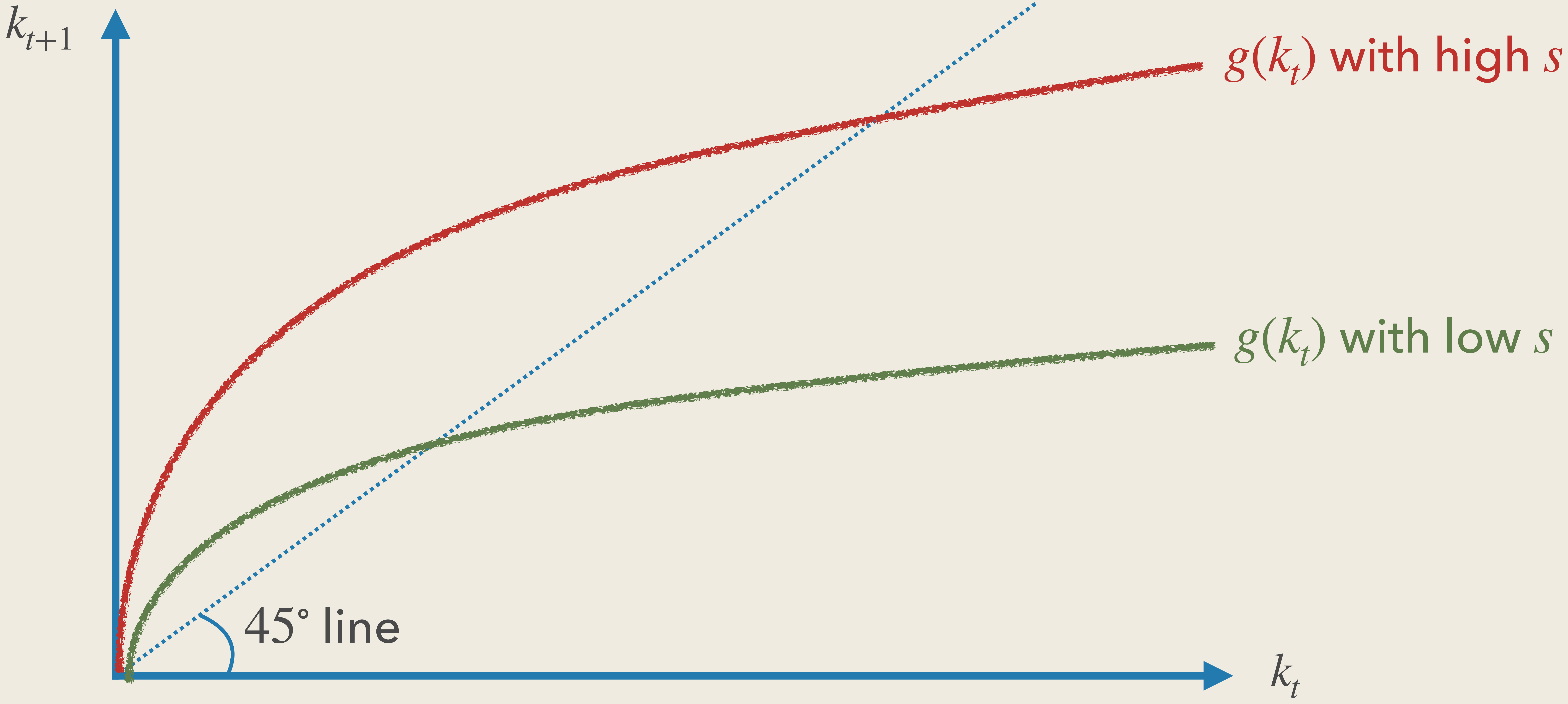

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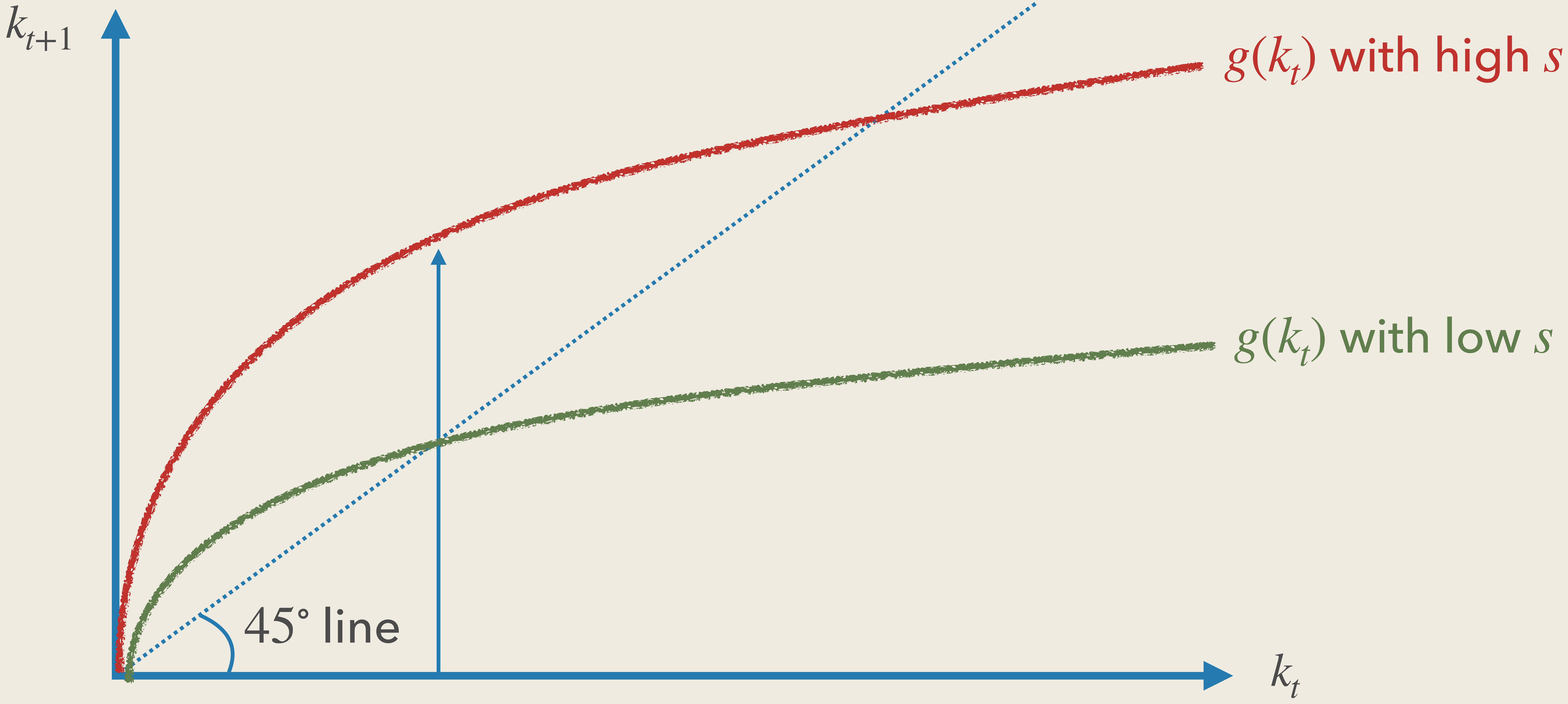

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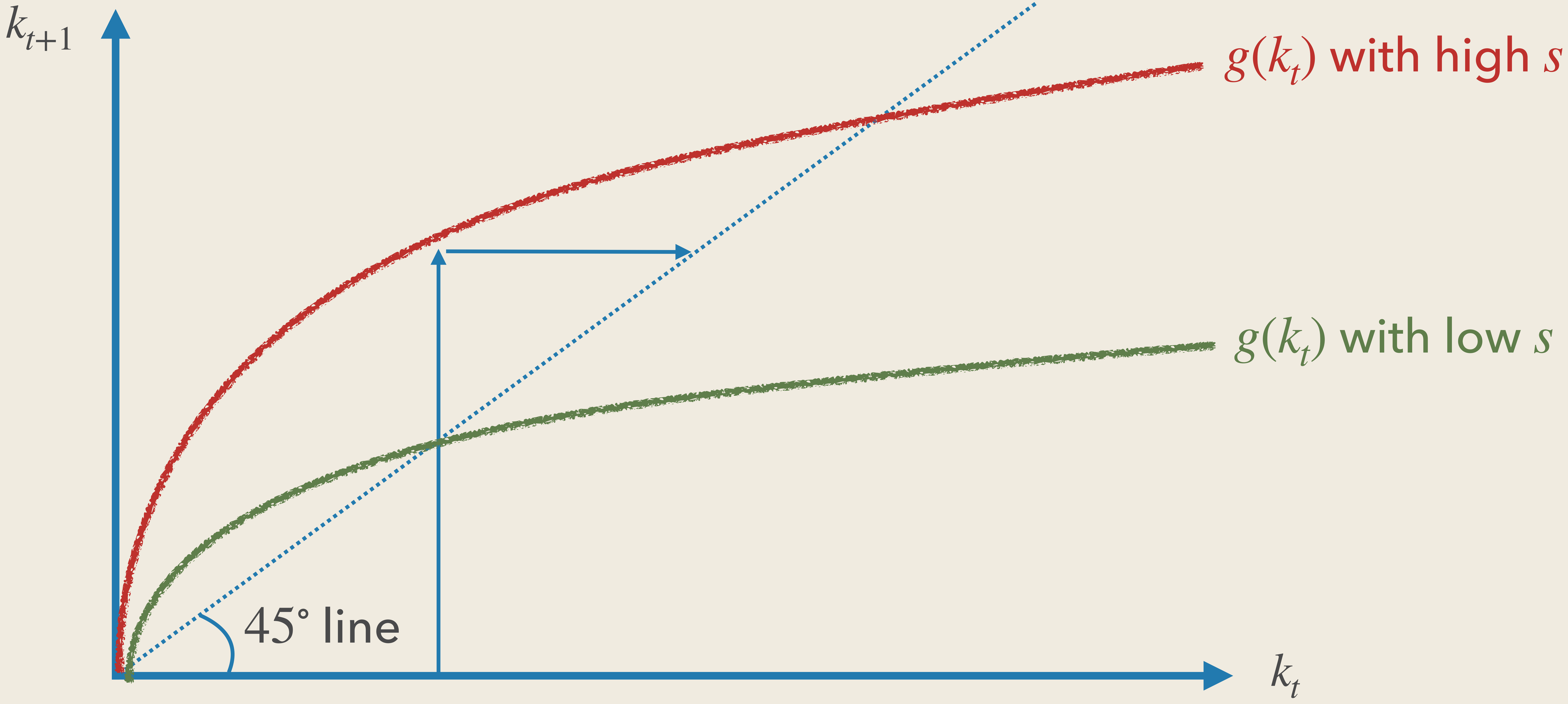
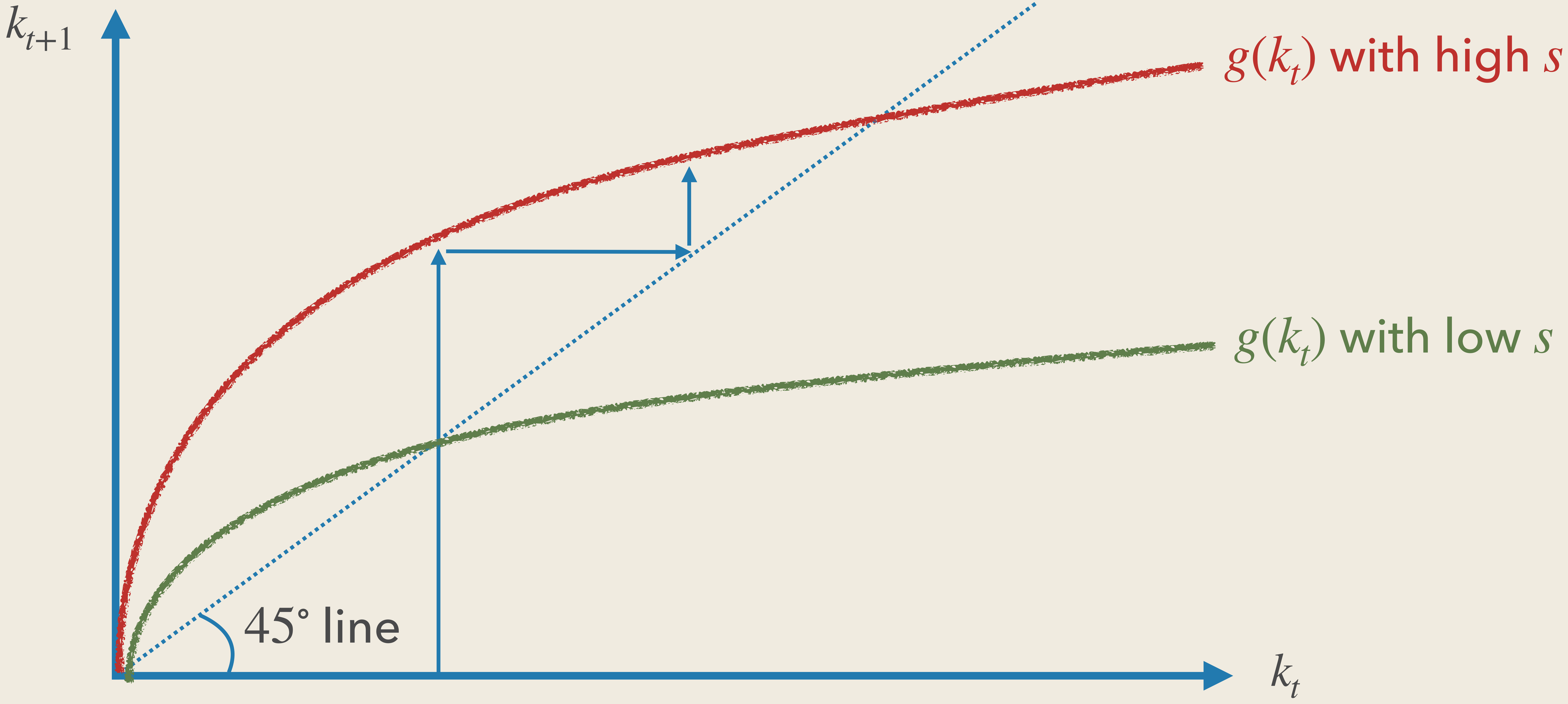
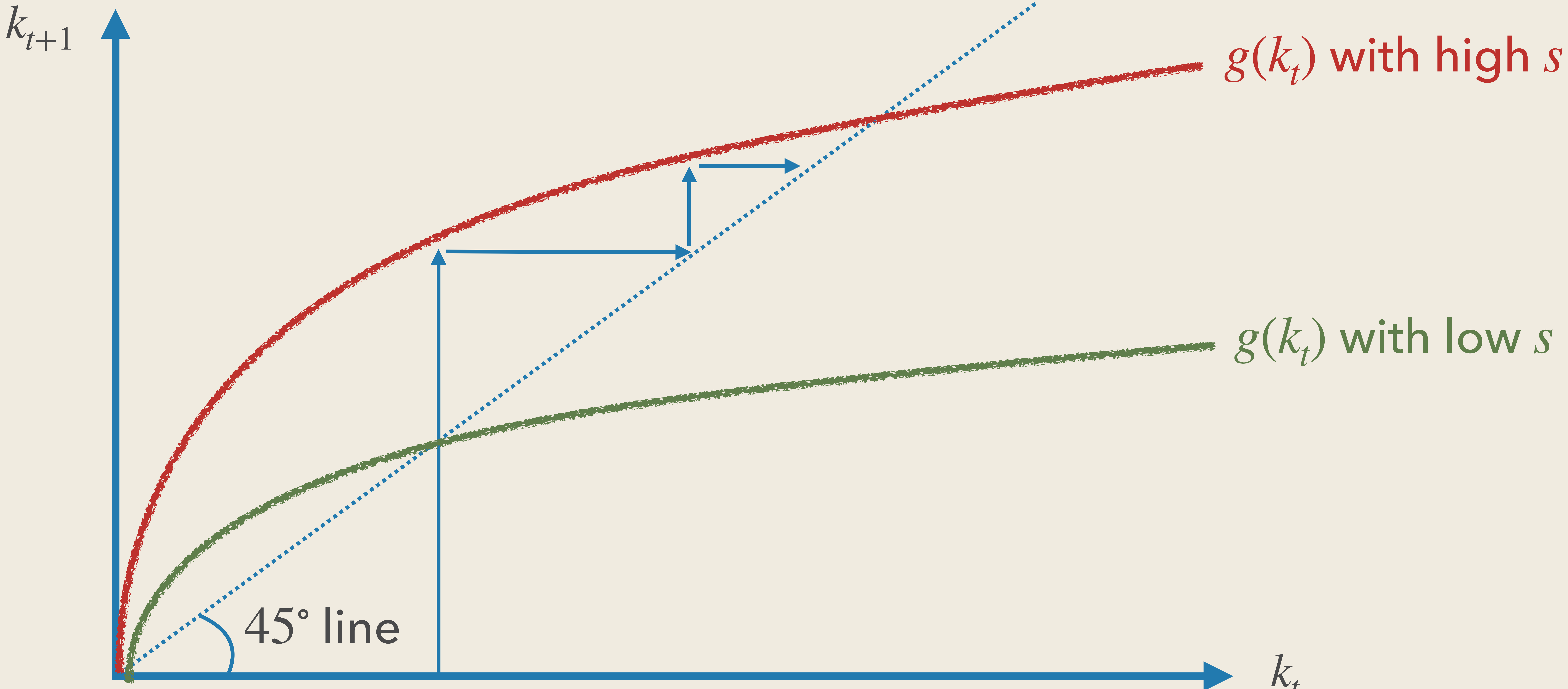


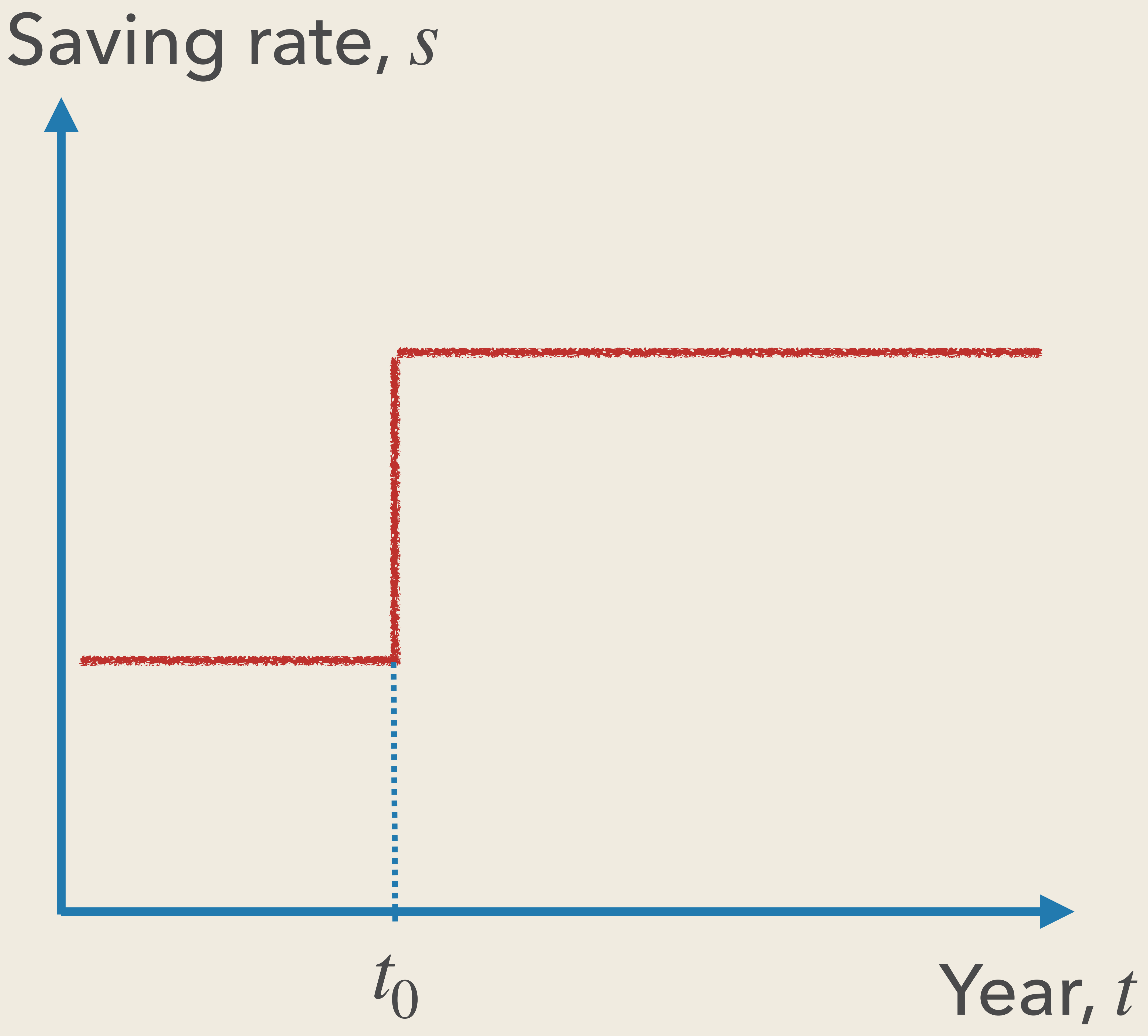
Evolution of Capital Stock

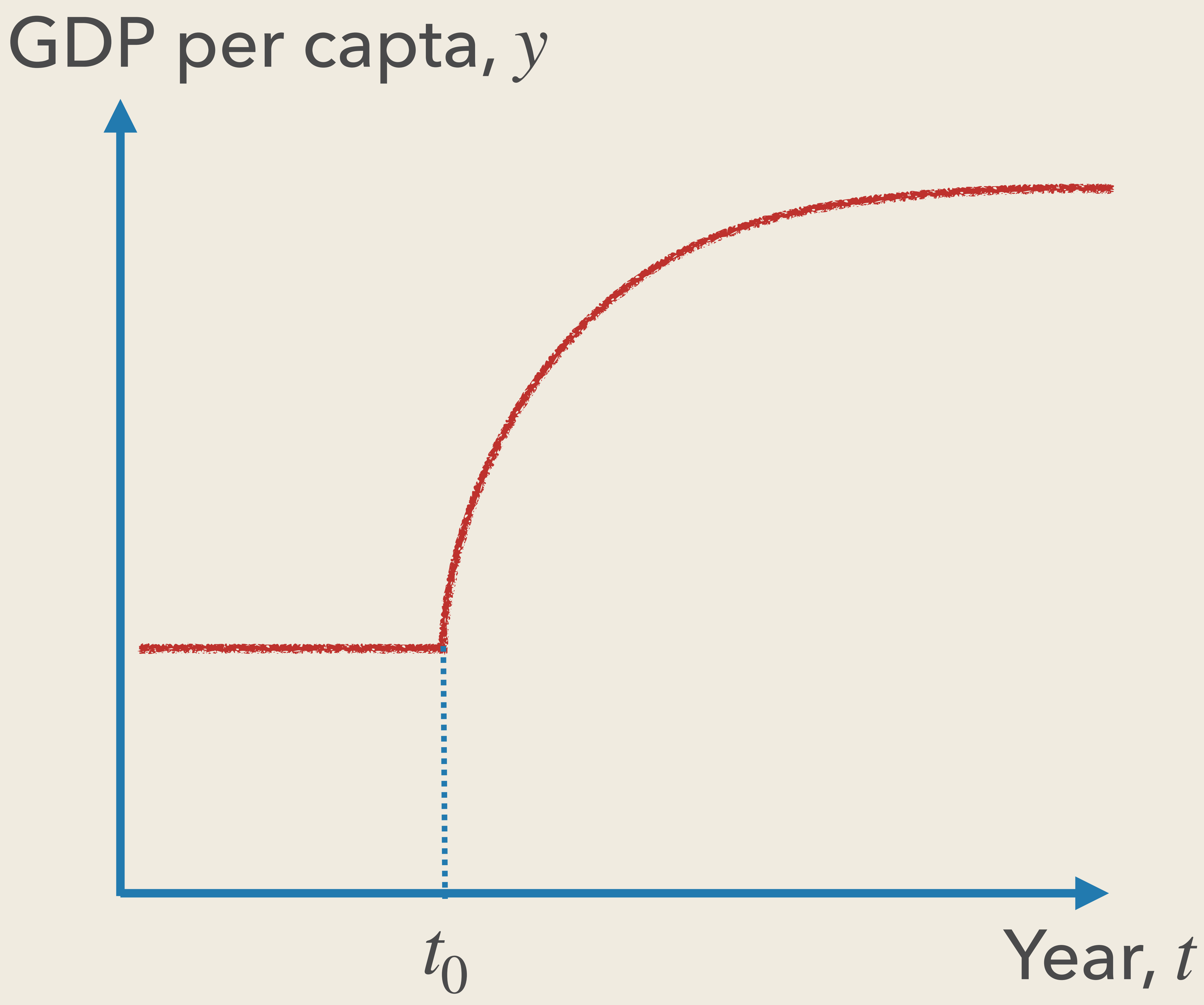


Evolution of Capital Stock

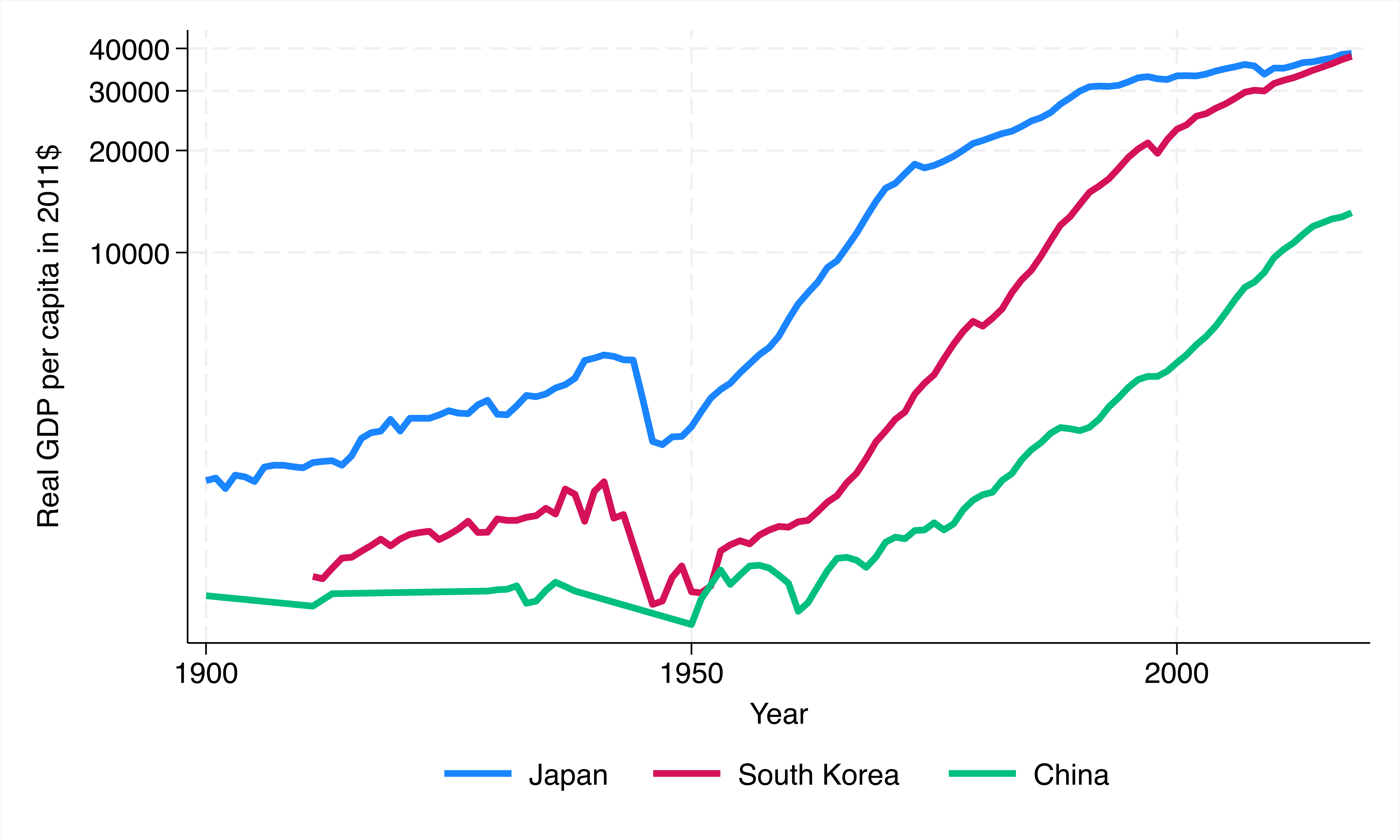


Growth Miracle?





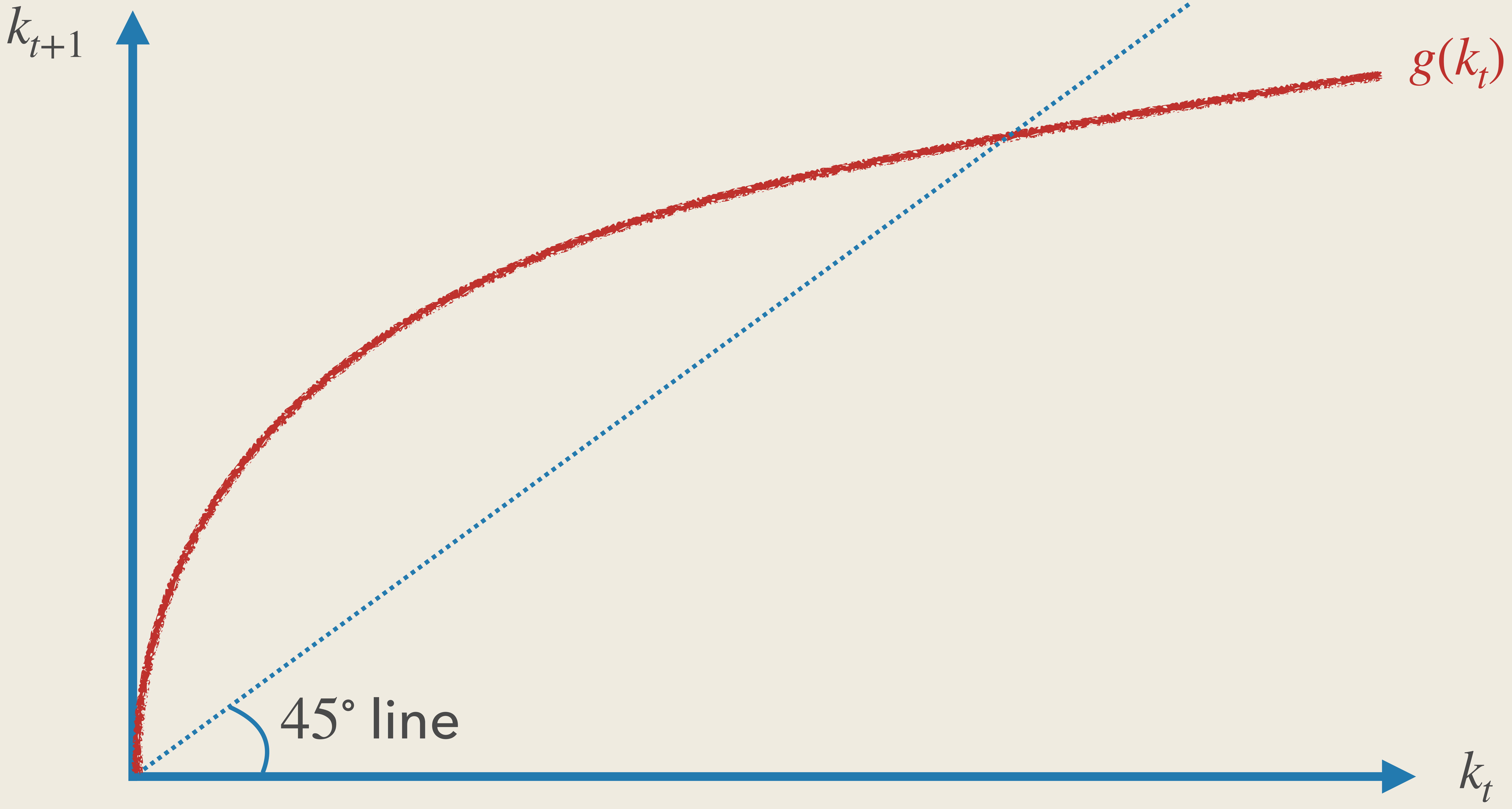
Asian Growth Miracle



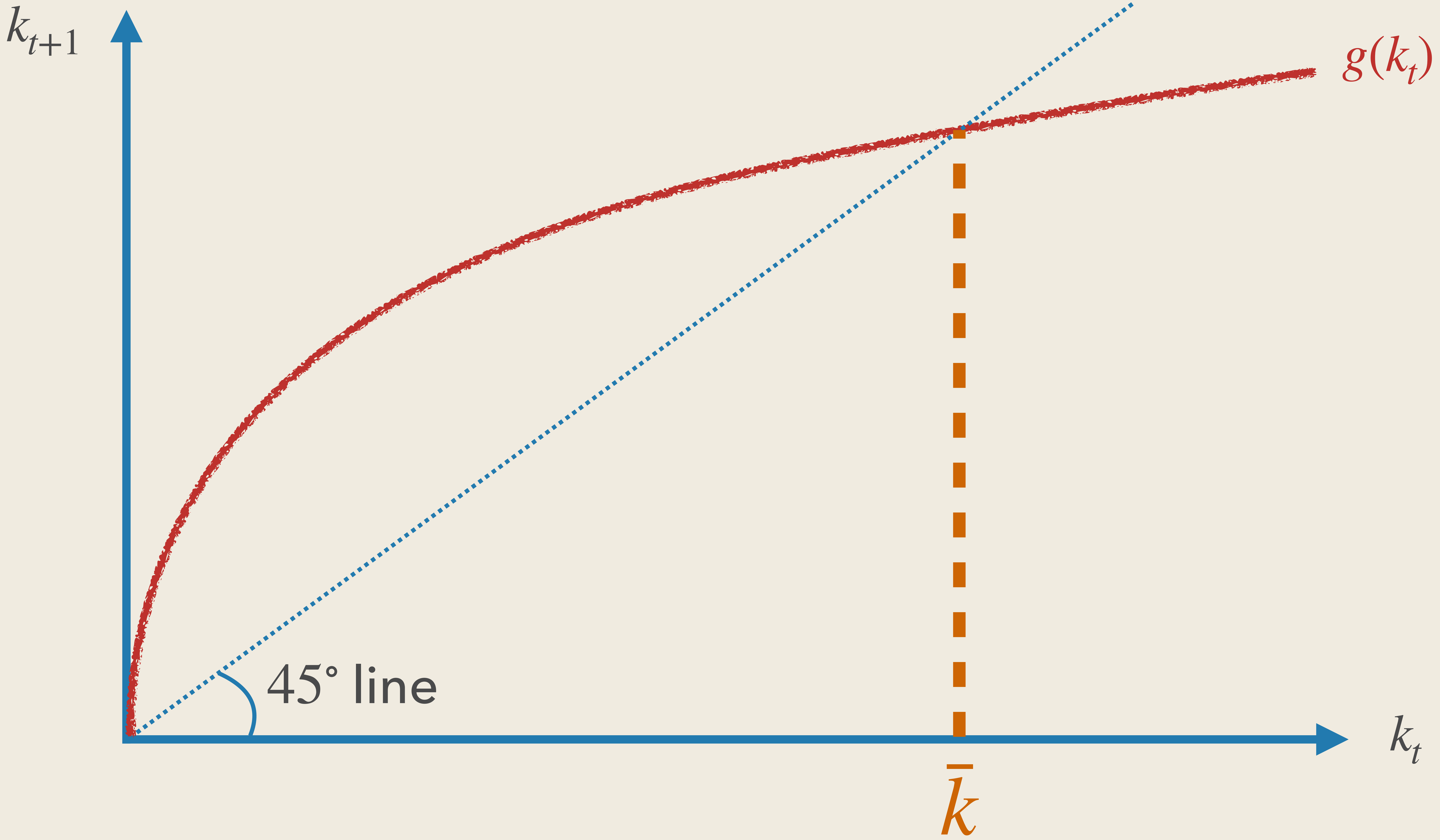
Capital Destruction

- Another interesting prediction of Solow model is capital destruction
- Suppose a country begins in a steady state
- What happens if some of its capital stock is suddenly destroyed?
  - due to wars or disasters

Evolution of Capital Stock



Evolution of Capital Stock



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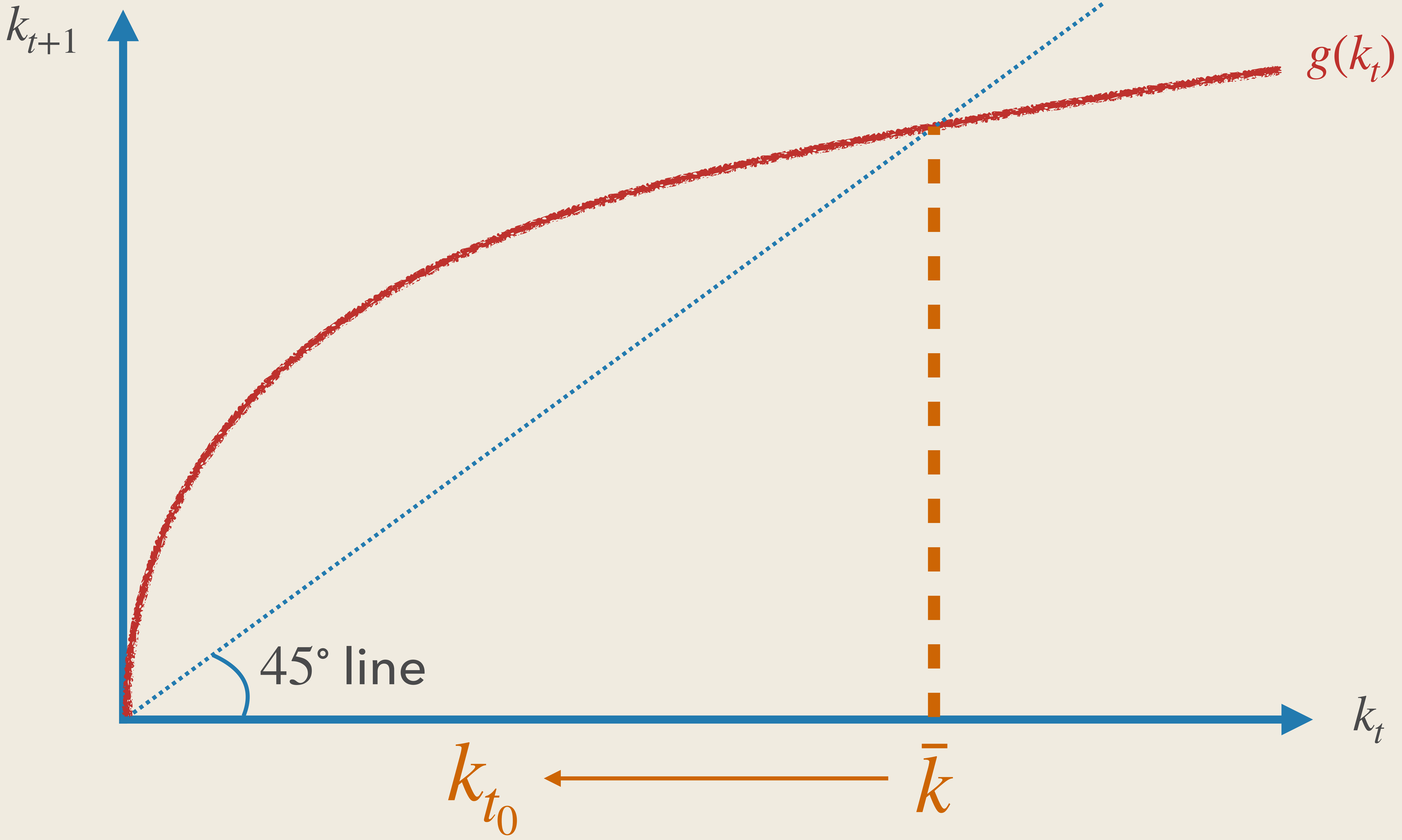

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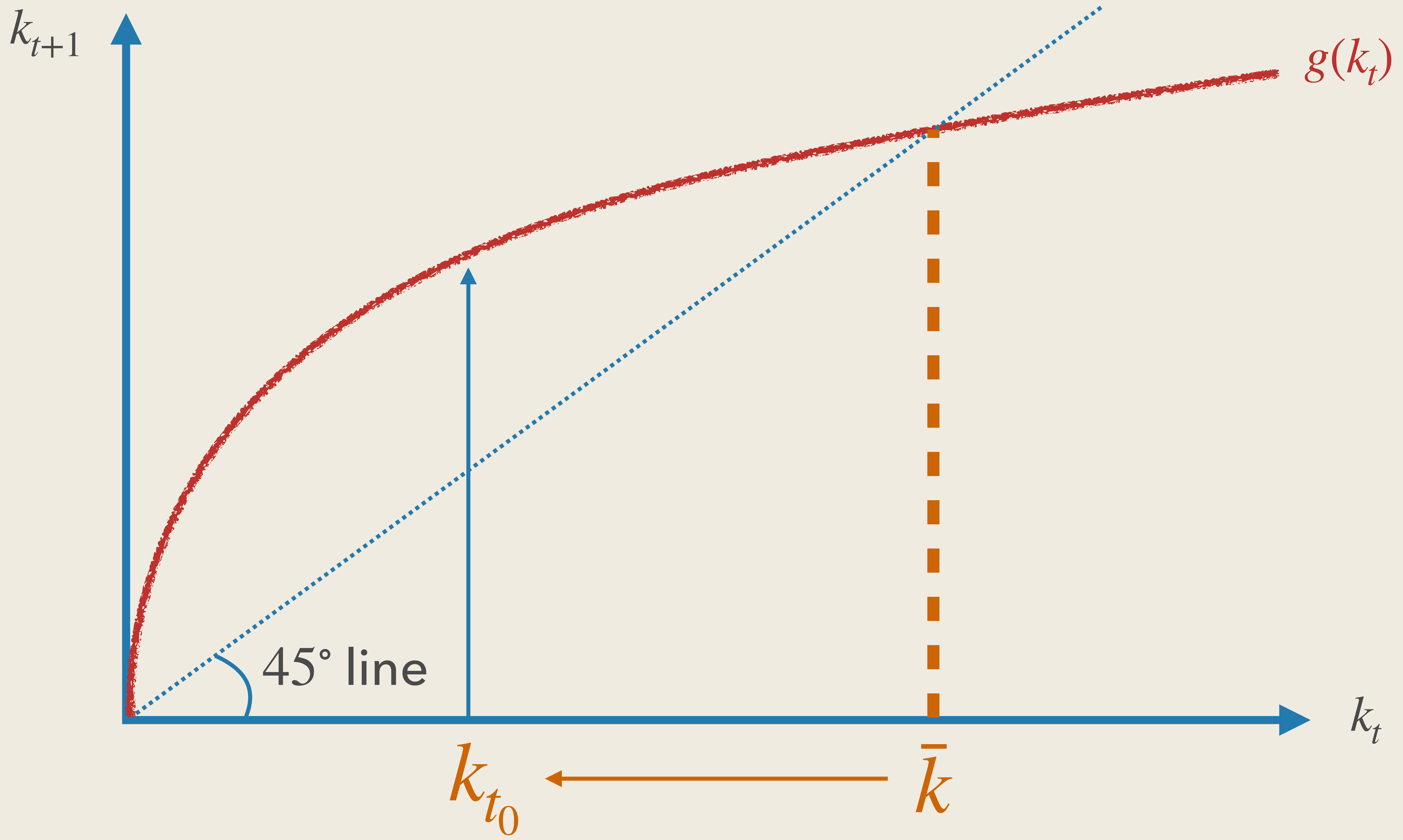

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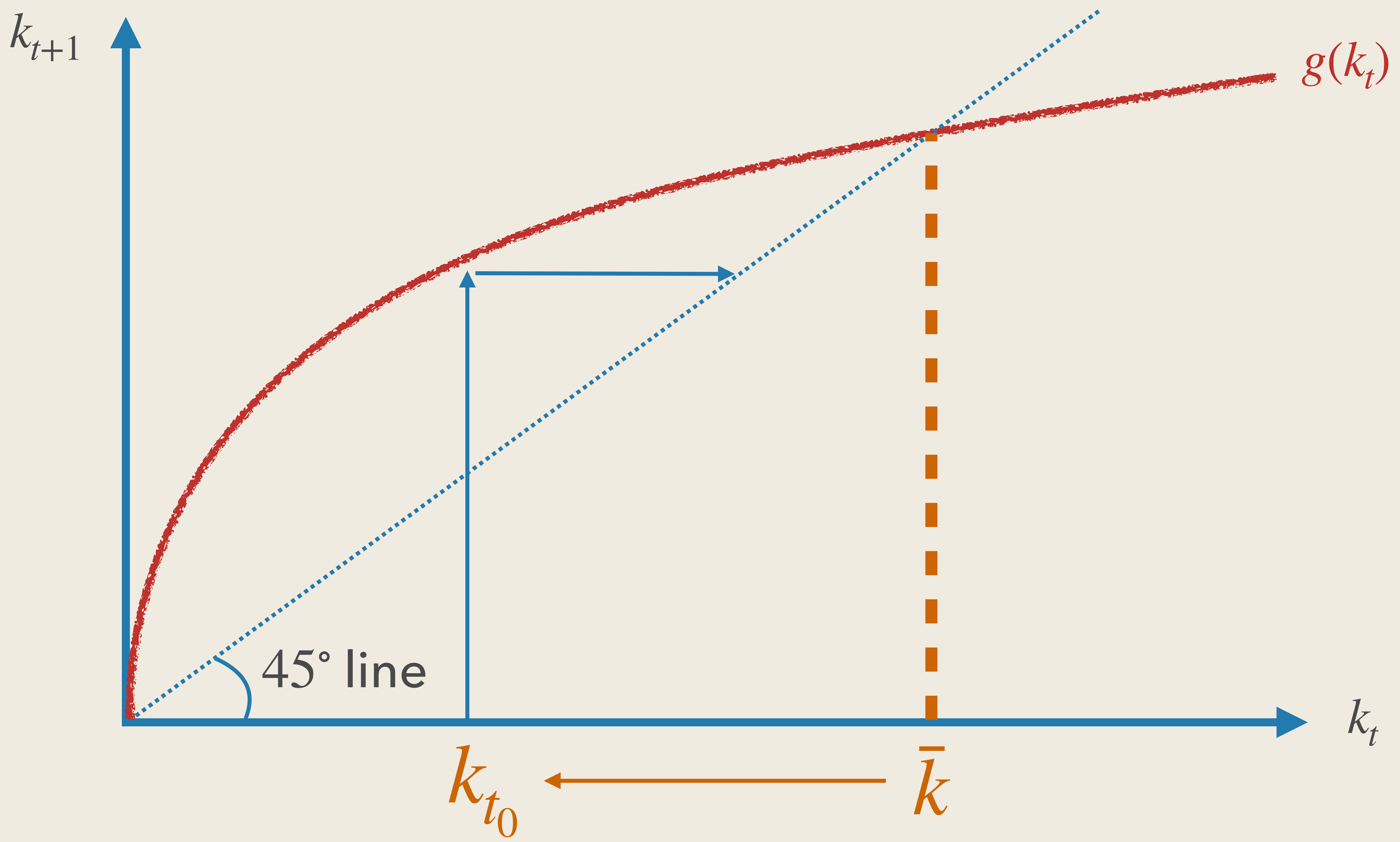

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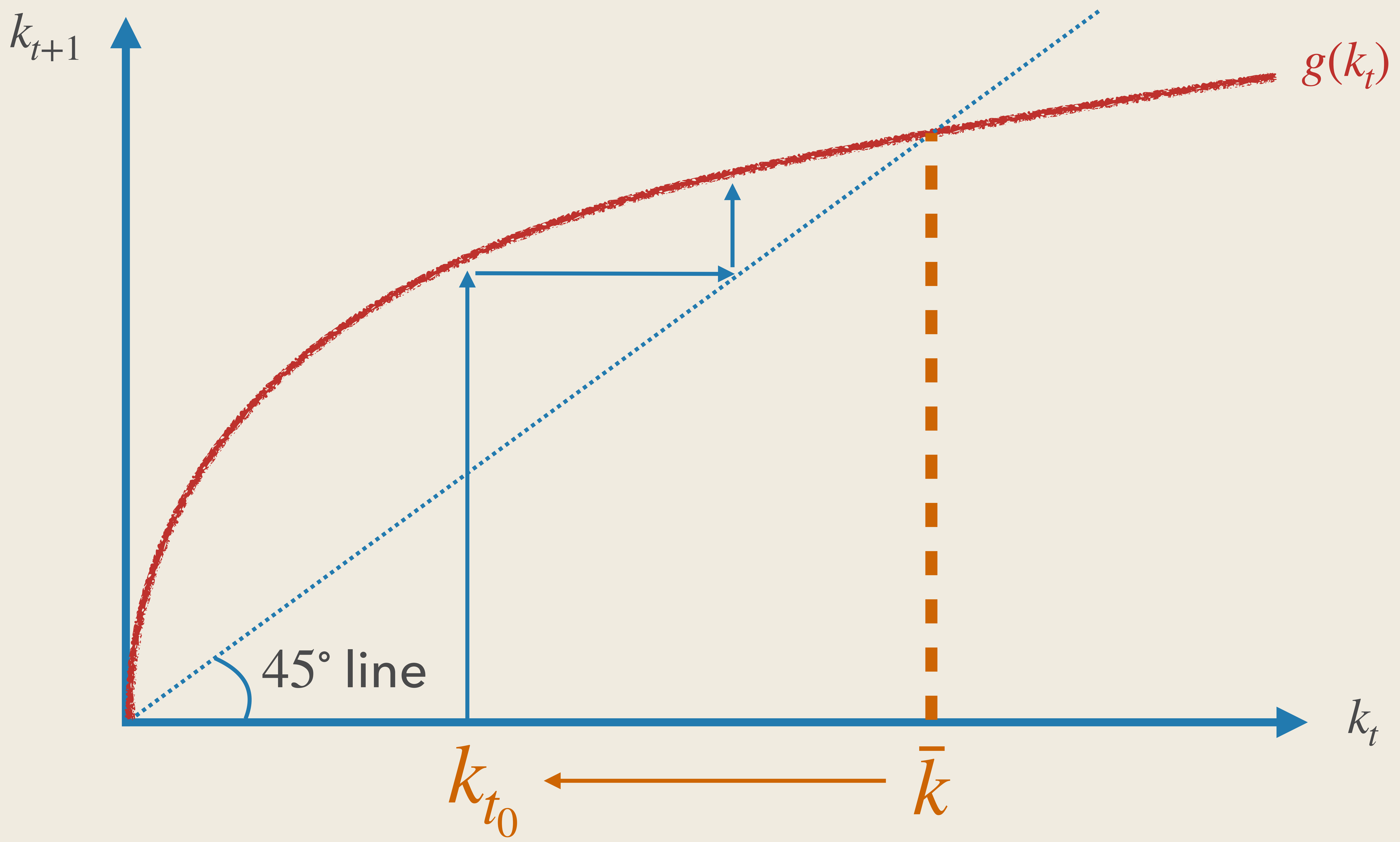

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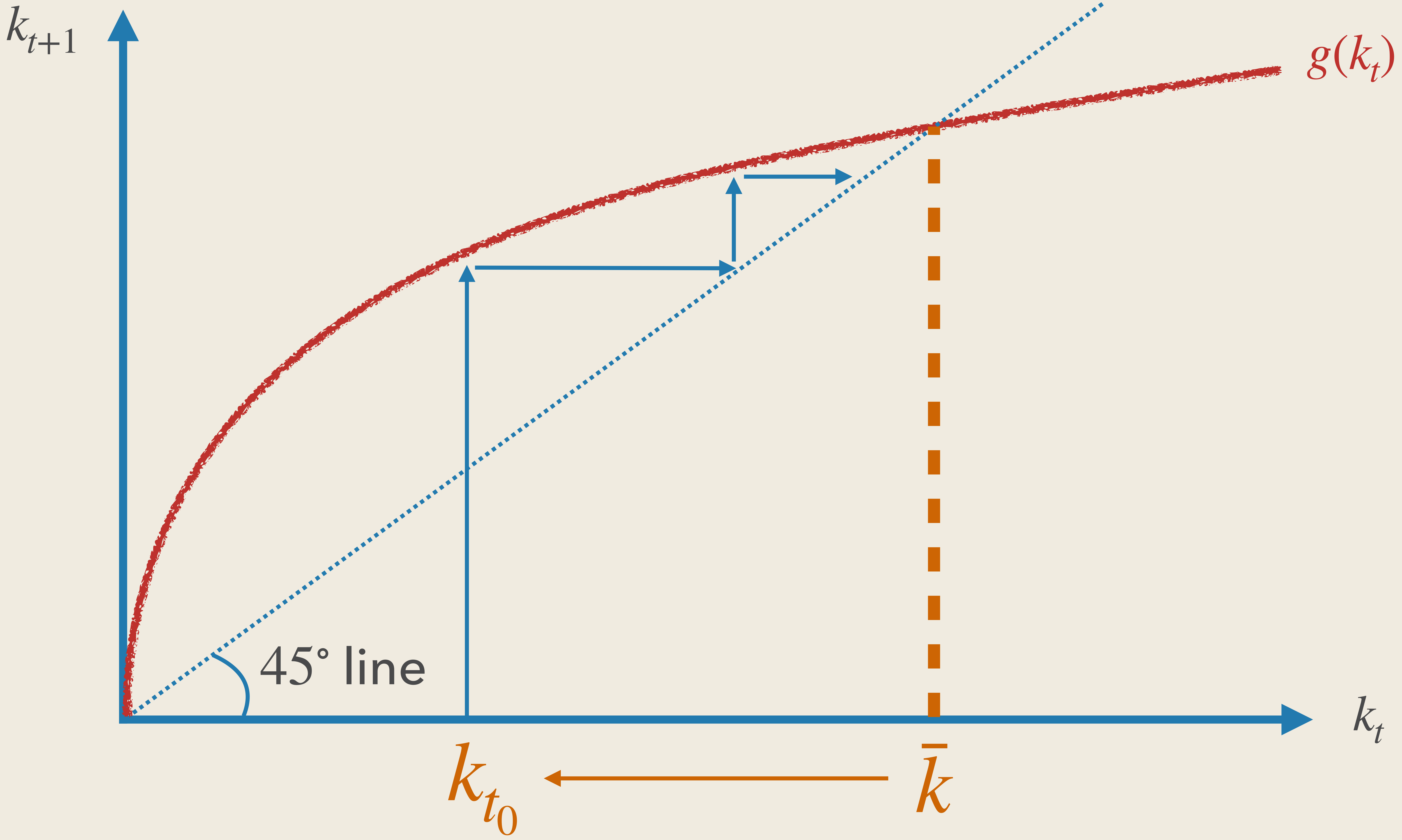
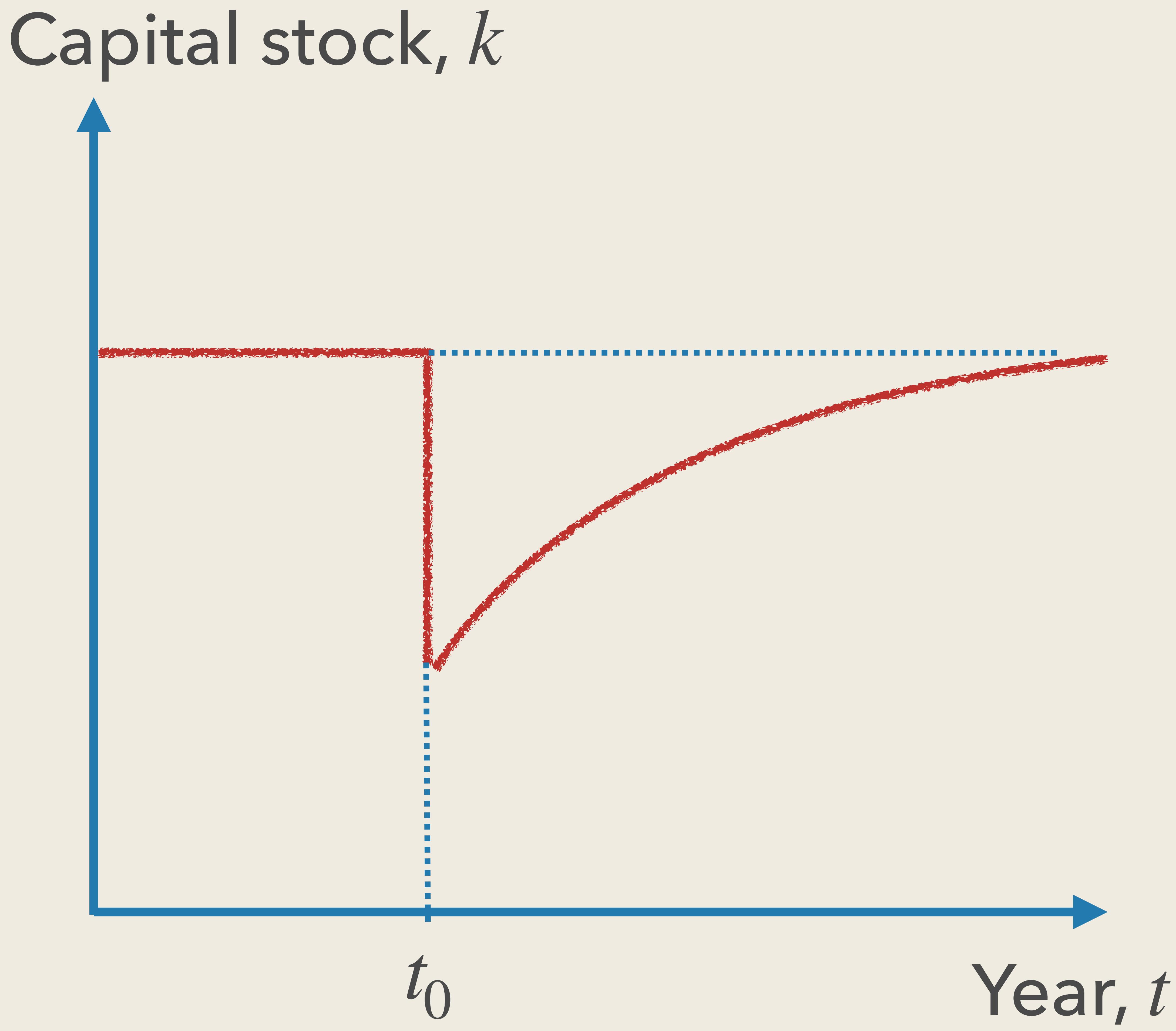
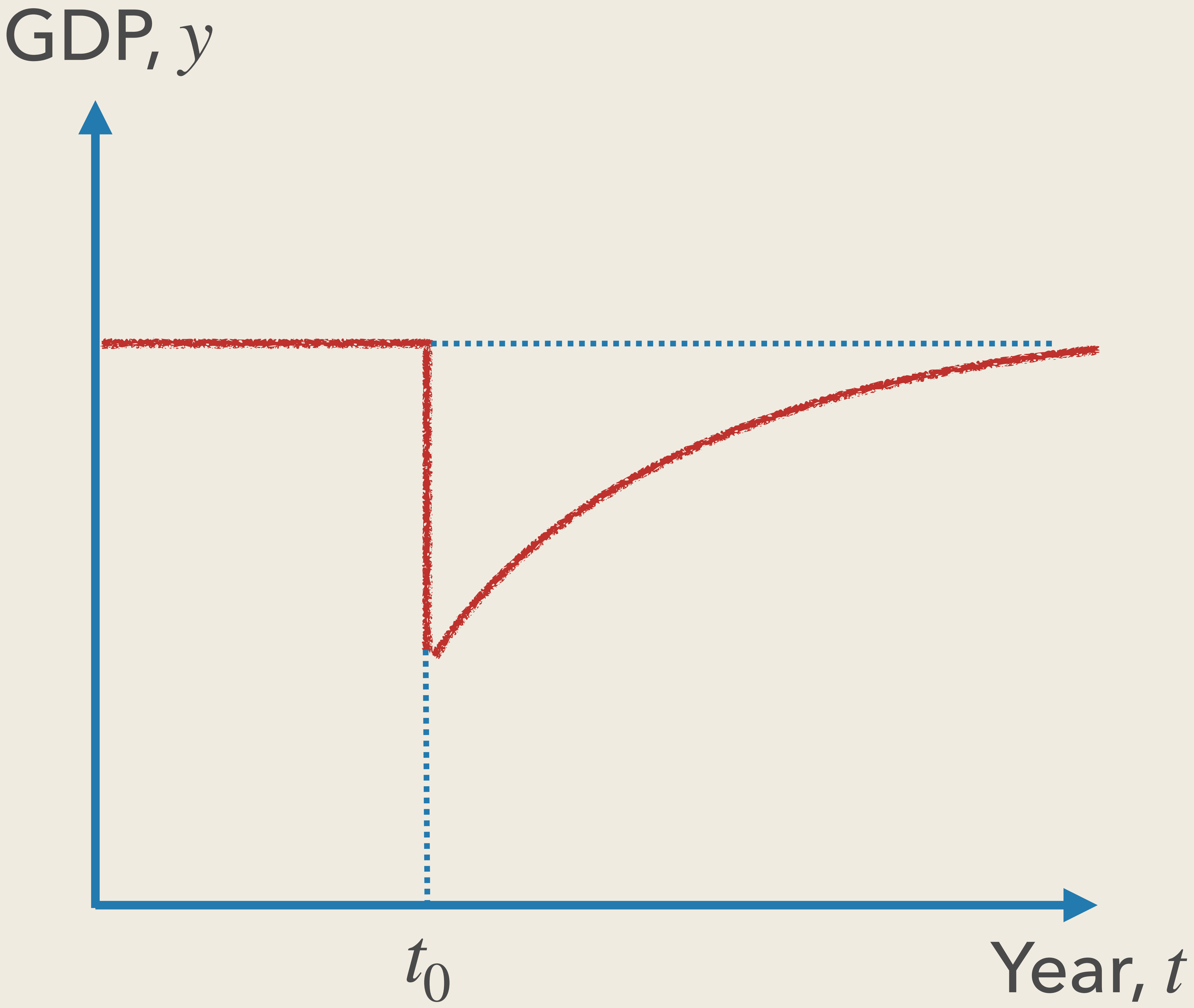


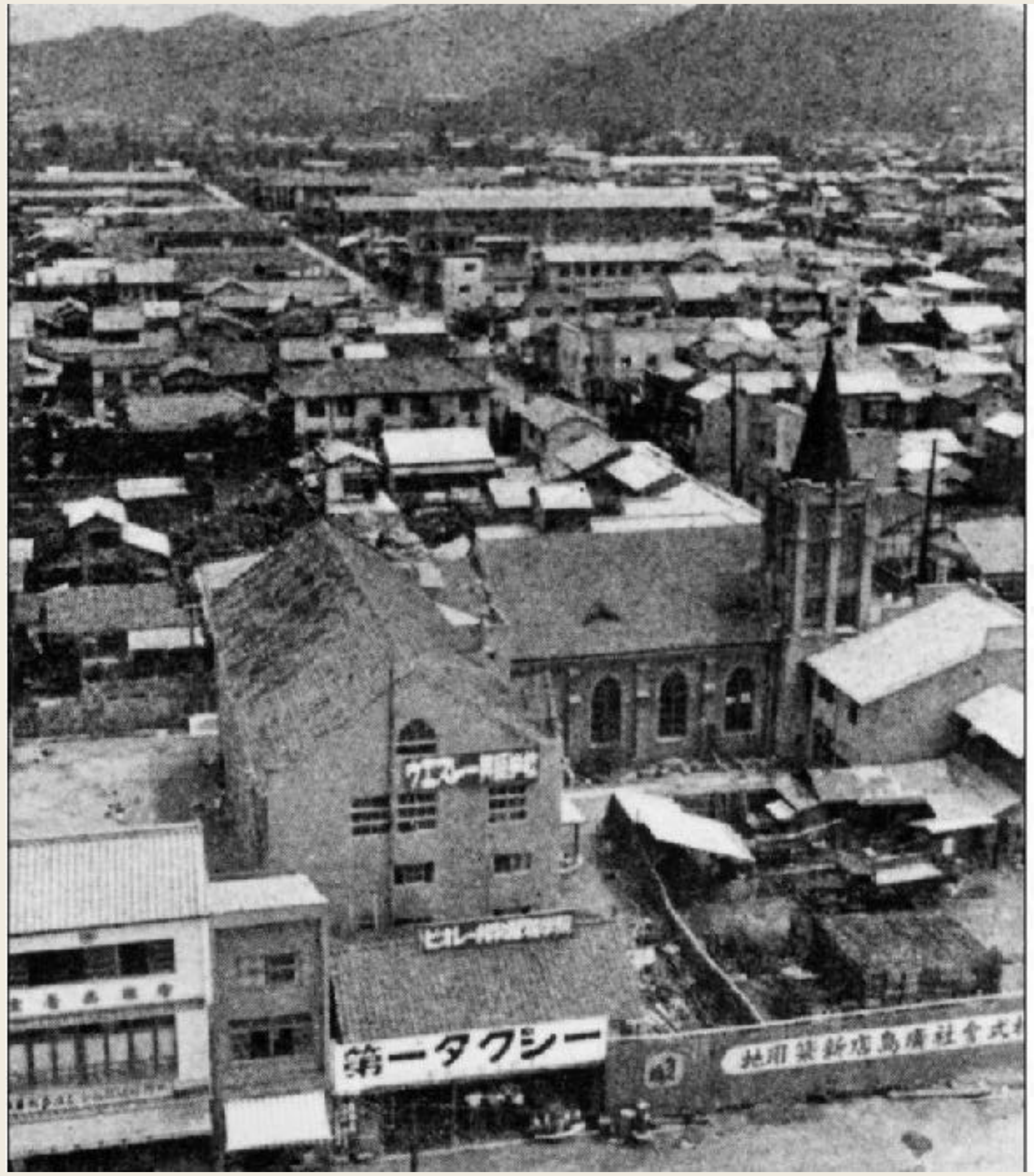
Capital Destruction Shock



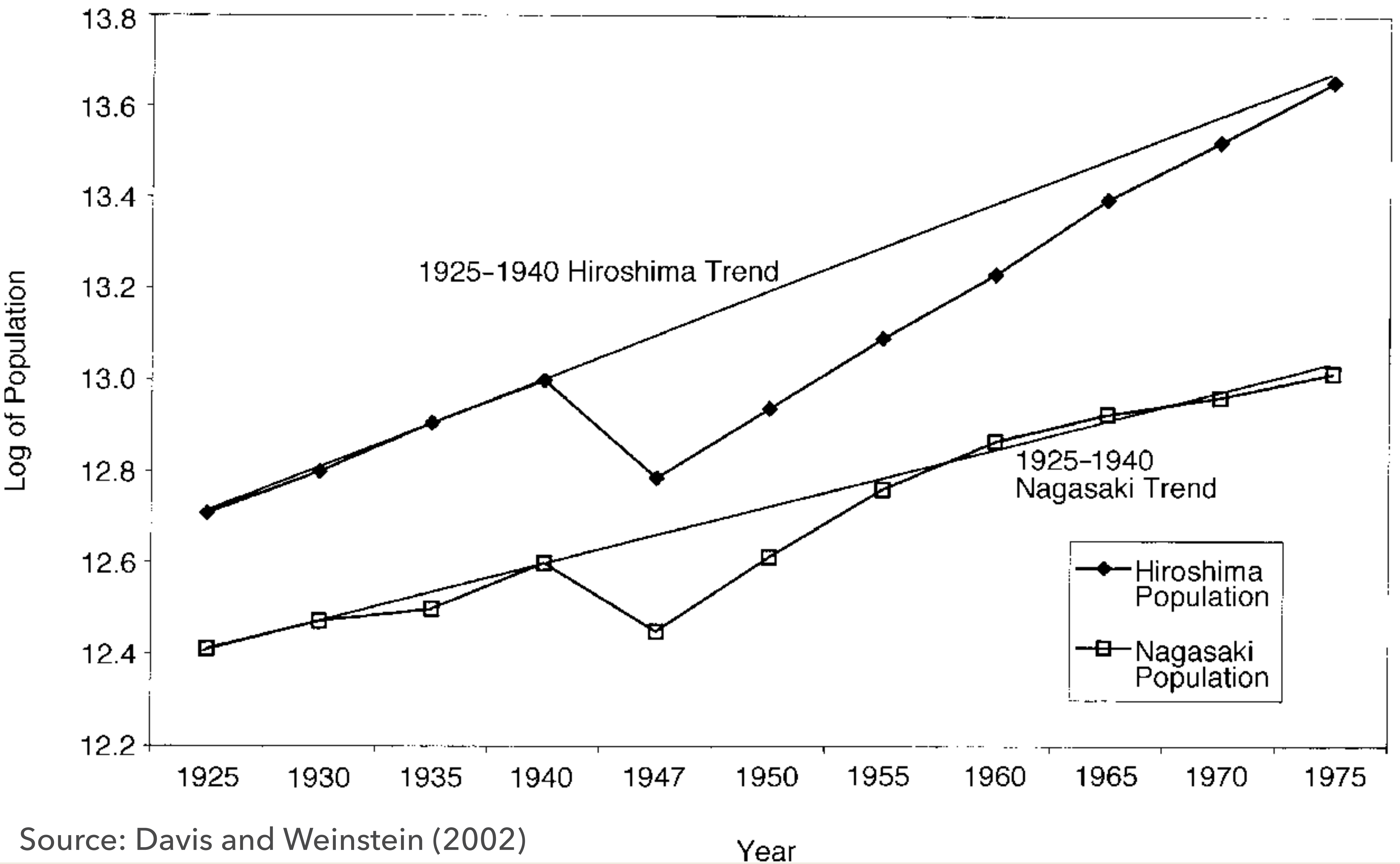


Davis and Weinstein (2002)

- Davis and Weinstein (2002):
  test this prediction using atomic bombing of Hiroshima and Nagasaki as a laboratory



Rapid Recovery after Bombing



Source: Davis and Weinstein (2002)

Nagasaki 1945 and Today

Source: https://www.theguardian.com/artanddesign/gallery/2015/aug/06/after-the-atomic-bomb-hiroshima-and-nagasaki-then-and-now-in-pictures

Can Investment be Too High?

Investment Too High or Too Low?

- High saving (investment) rates are the source of capital accumulation
- Should the investment rates be high? Can it be too high?
- Think of an extreme example with $s = 1$
  $\Rightarrow$ You consume nothing because $c = (1 - s)y = 0$
- Then, should the investment rate be low?
- Think of an extreme example with $s = 0$ and recall $\bar{k} = \left(As/(n + \delta)\right)^{\frac{1}{1-\alpha}}$ in the long-run
  $\Rightarrow$ Again, you consume nothing in the long-run because $c = (1 - s)\bar{y} = (1 - s)A\bar{k}^{\alpha} = 0$

Golden Rule of Saving Rate

- So what is the investment rate that maximizes long-run per-capita consumption?
- Steady-state (long-run) consumption is given by

$$c(s) \equiv (1-s)A\left(\frac{As}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

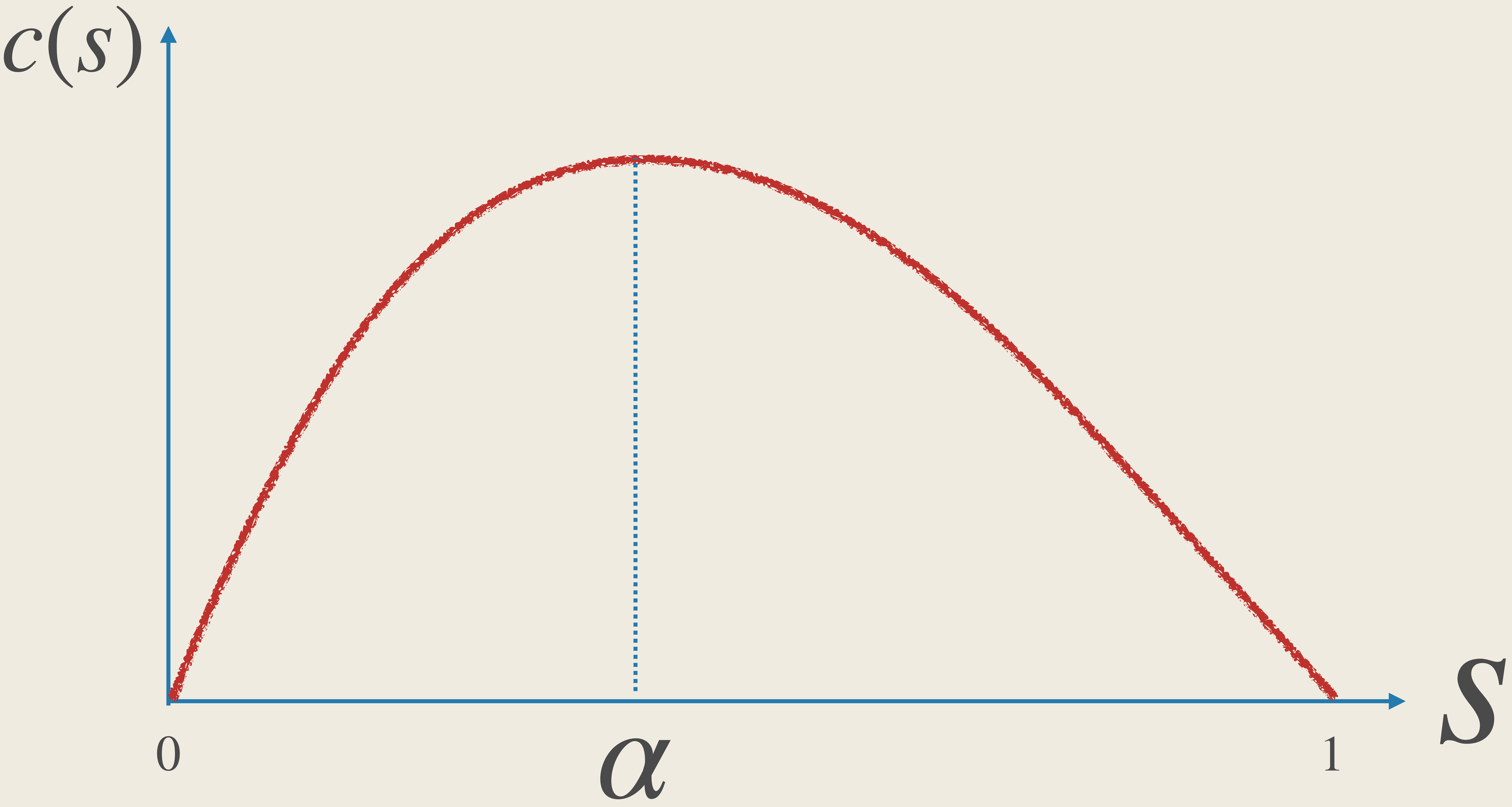
- The saving rate that maximizes the steady-state consumption, s^*, solves

$$\max_s c(s)$$

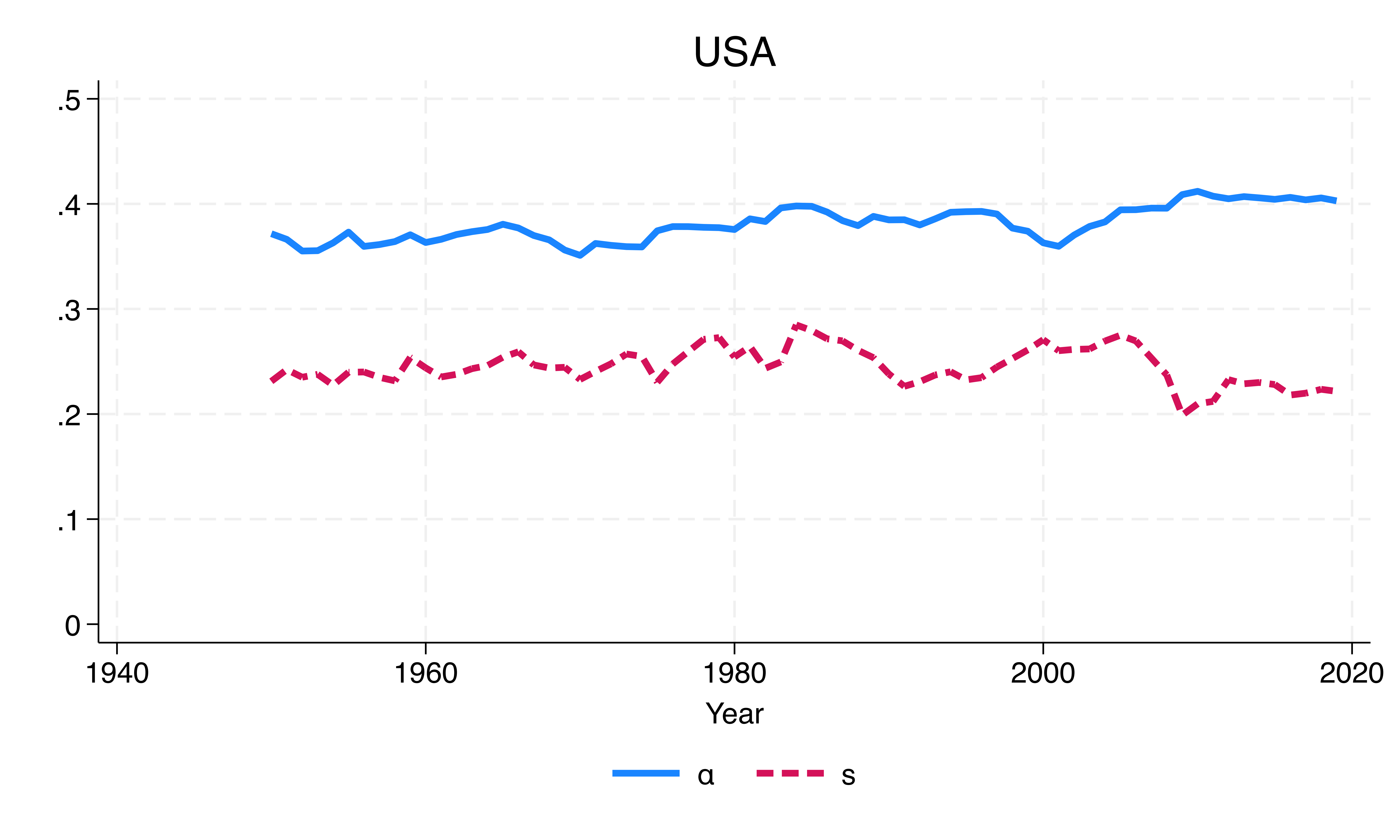
- Taking the first-order condition,

$$\frac{dc(s)}{ds} = \frac{\alpha - s}{(1-\alpha)s}A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

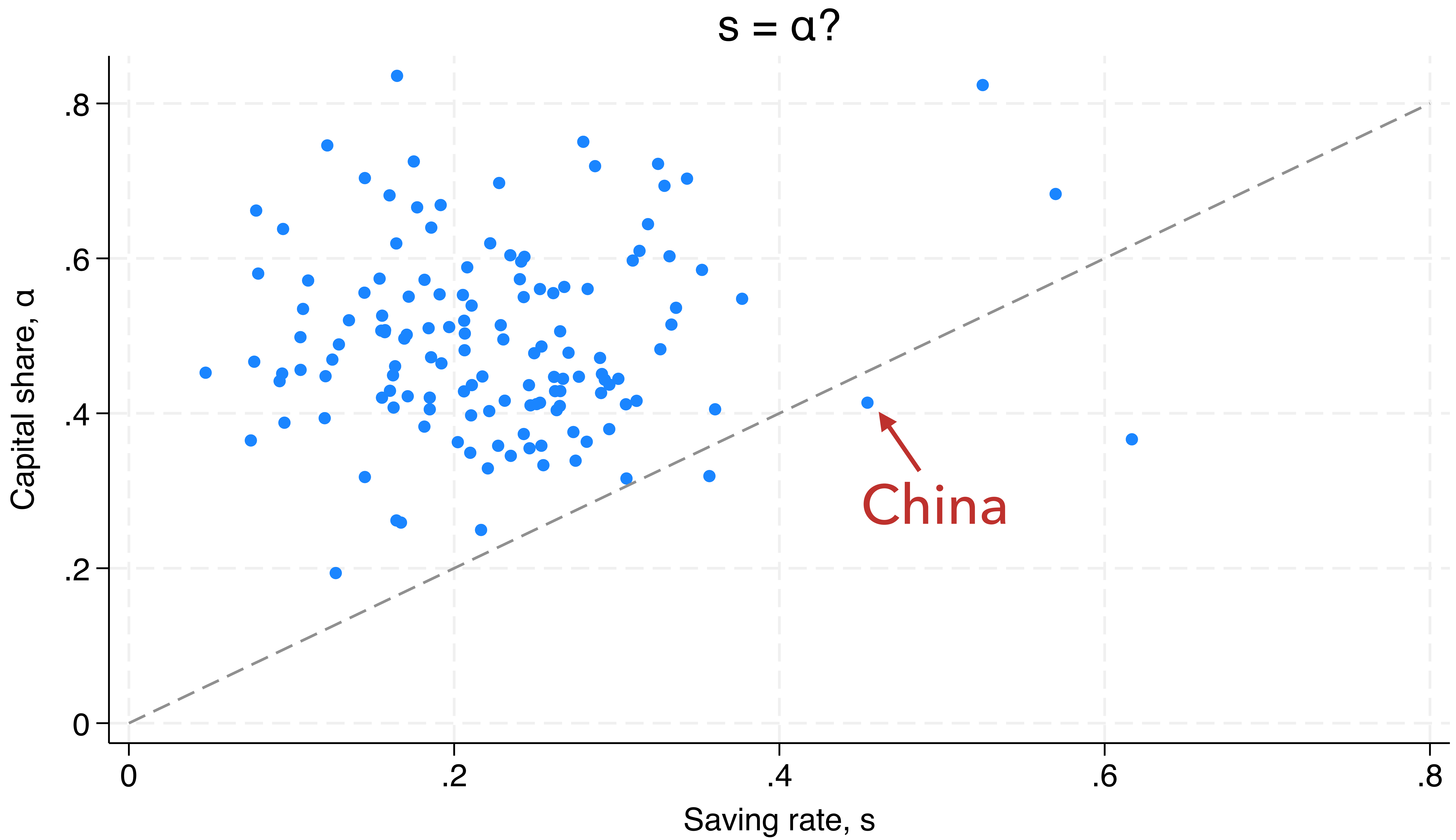
What Saving Rate Maximizes SS Consumption?



$s = \alpha$?



Cross-Country Data



Source: Penn World Table 10.01

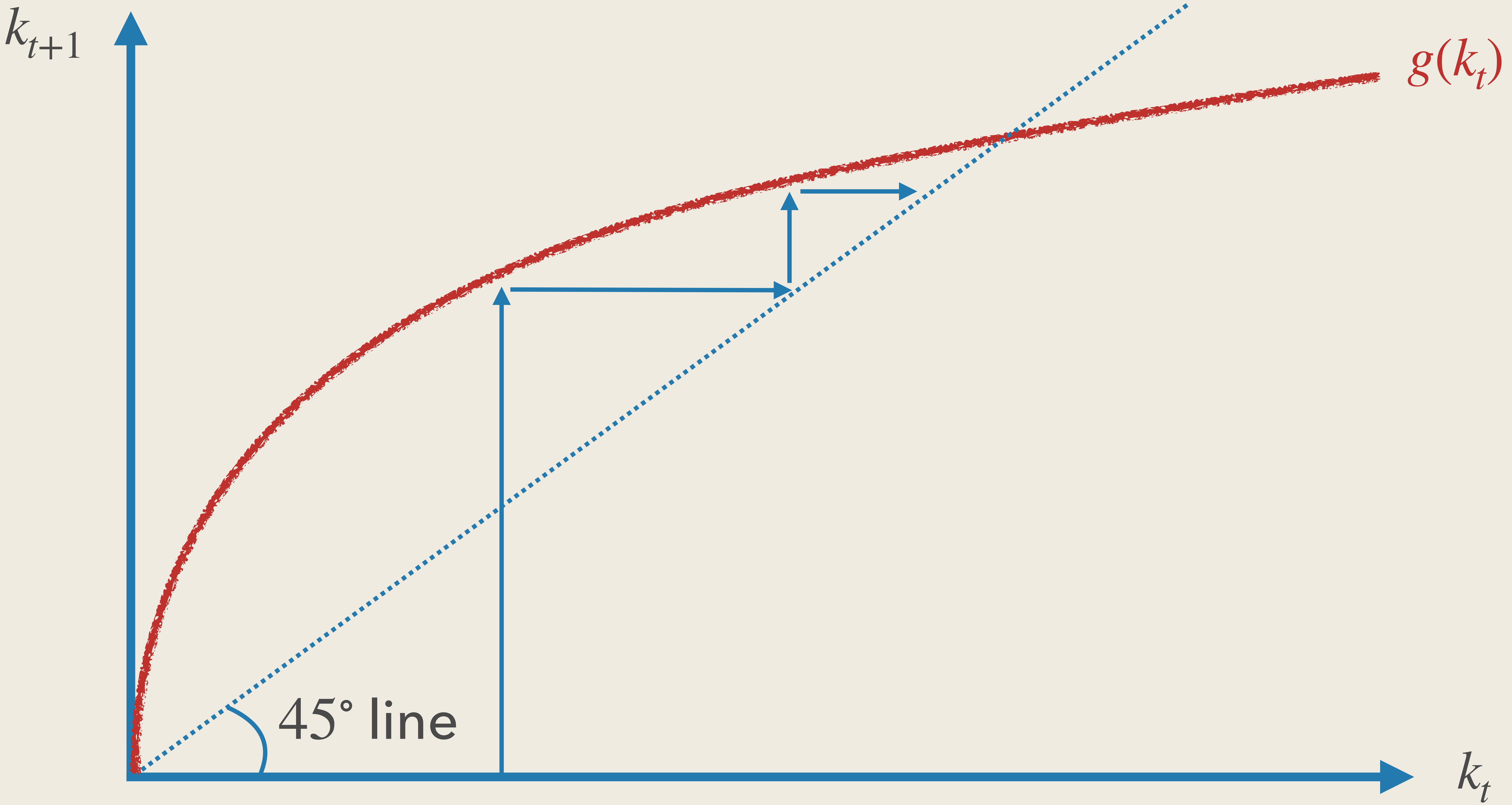
Caveat

- The golden rule of saving rate only concerns the steady state consumption
- It is not necessarily optimal from a welfare perspective
- Households may not care about steady state
- Remember, “in the long run, we are all dead”

Implications of Solow Model

Implication of Solow Model

- Countries with lower capital grow faster… **holding everything else equal**



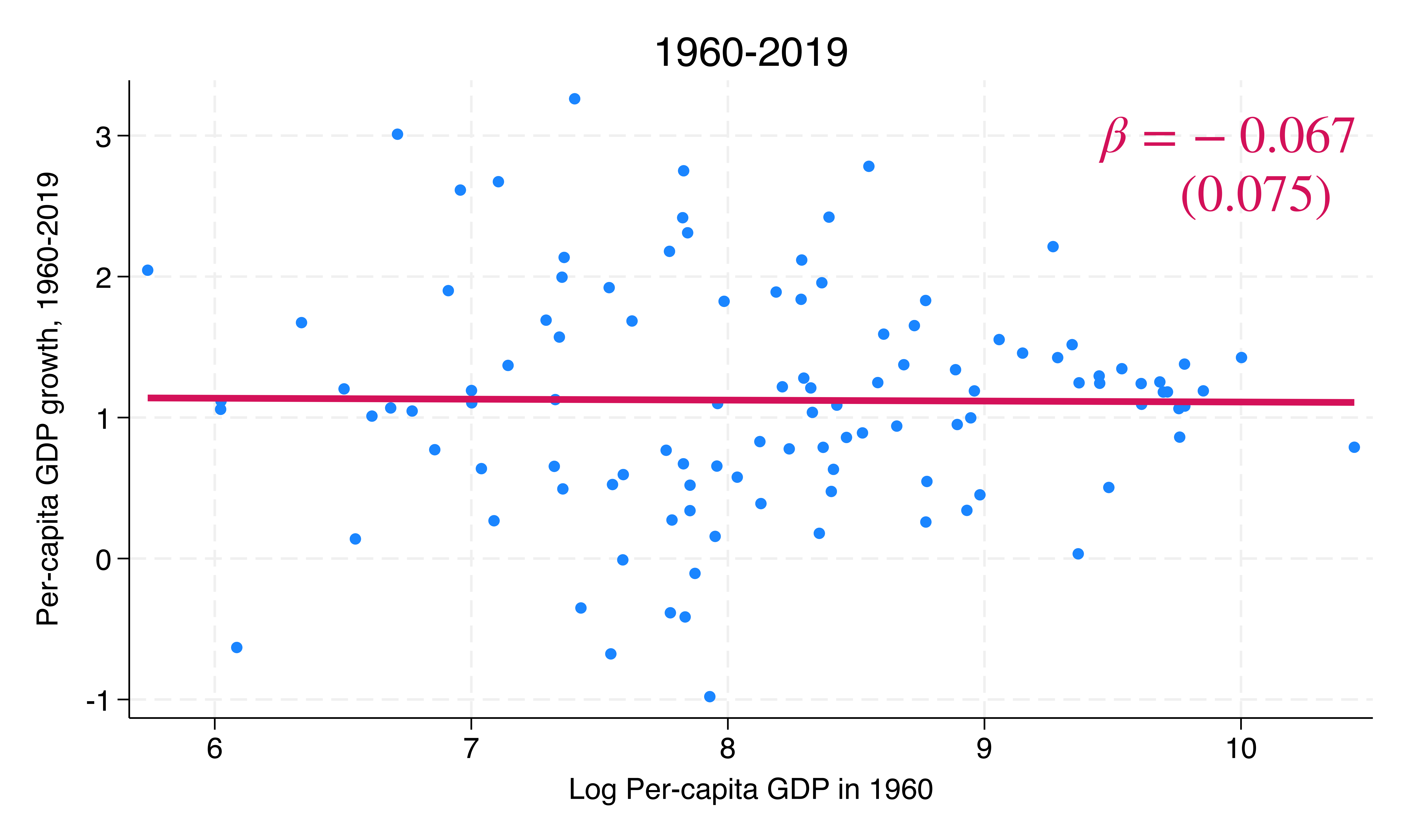
Testing Convergence

- Do initially poor countries grow faster subsequently in the data?
- Often called "unconditional convergence"
- Consider the following regression:

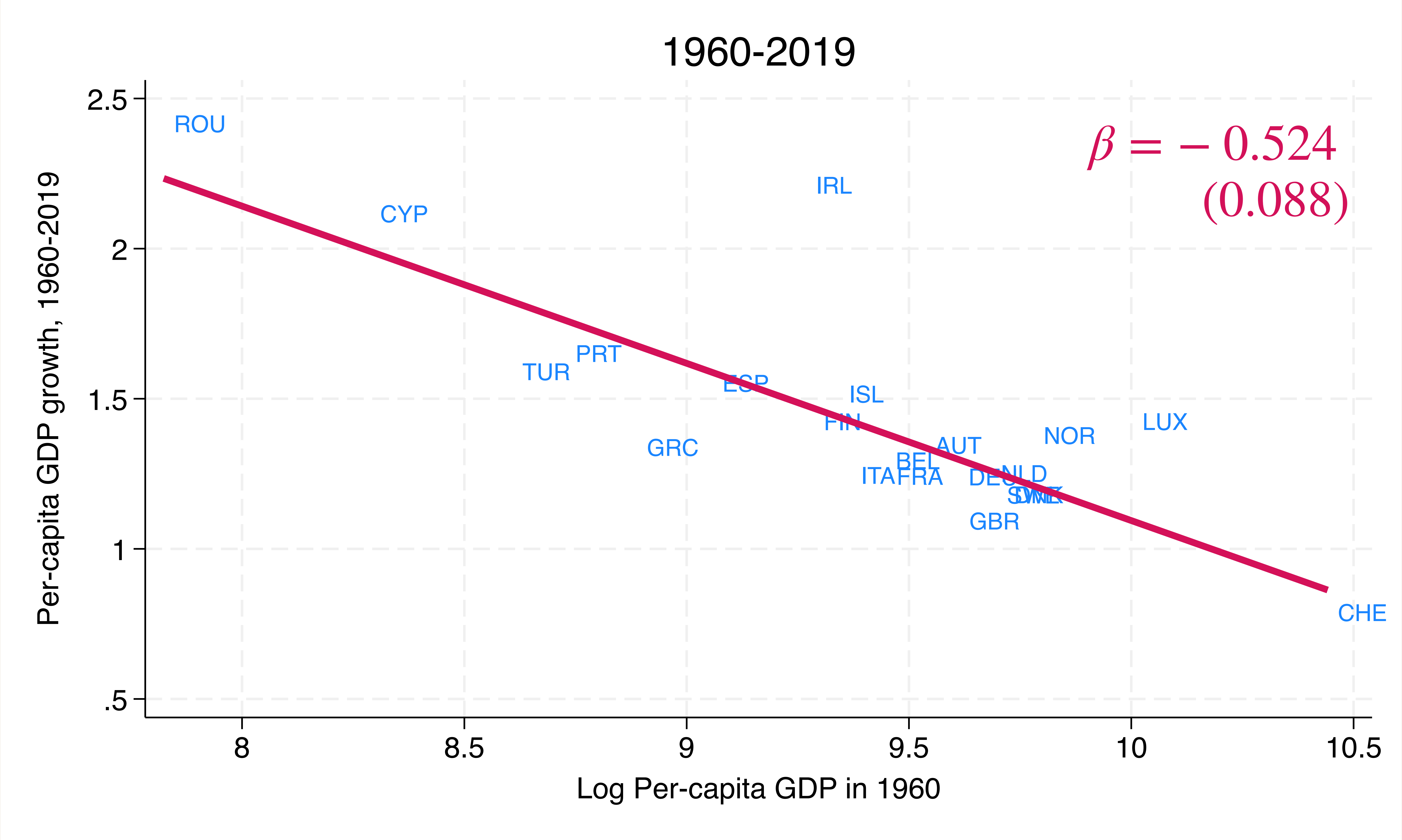
$$\log y_{i,t+T} - \log y_{i,t} = \gamma + \beta \log y_{i,t} + \epsilon_{i,t}$$

  - $\beta < 0$ implies that initially poor countries tend to grow faster

Convergence Regression

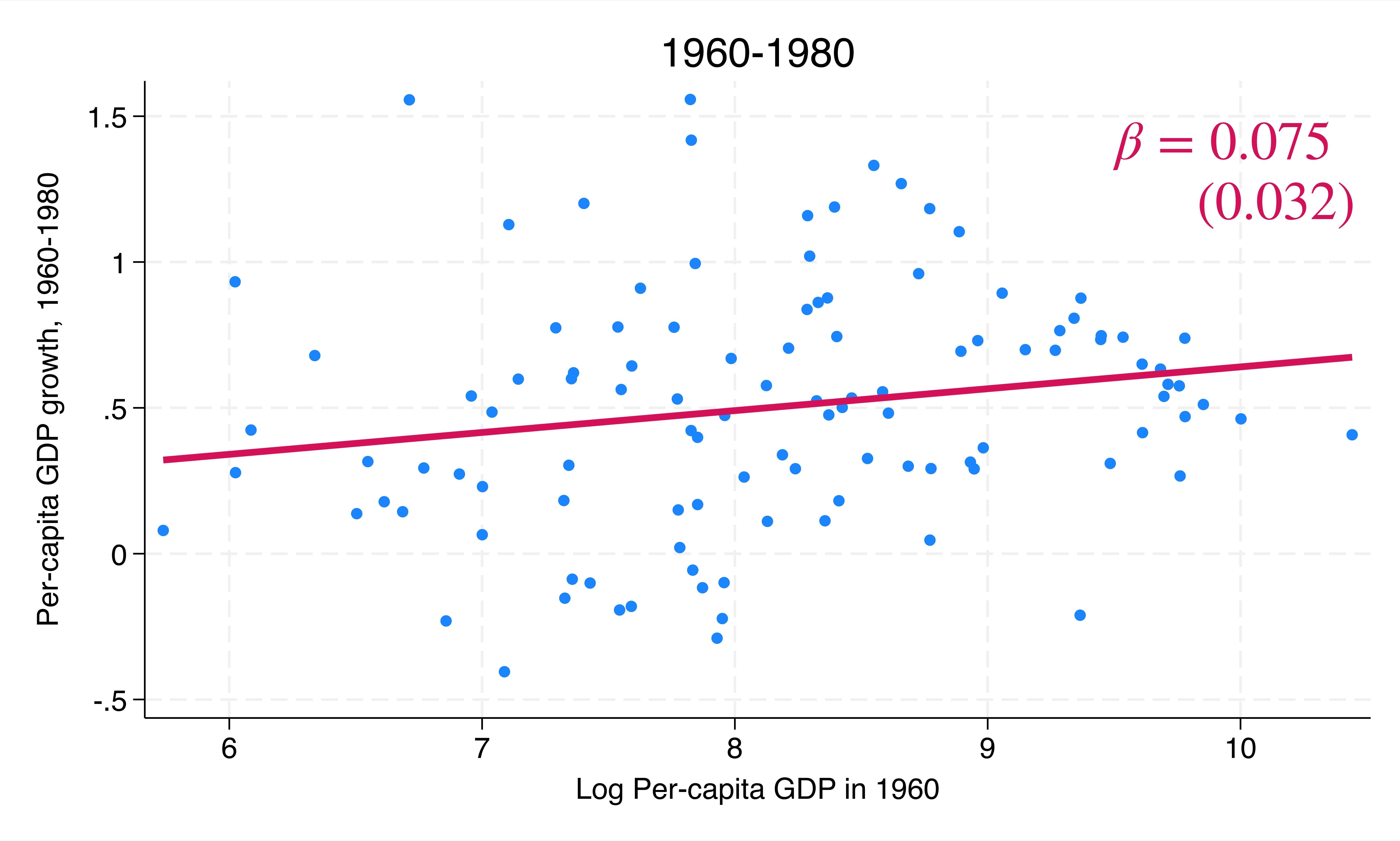


Only Europe

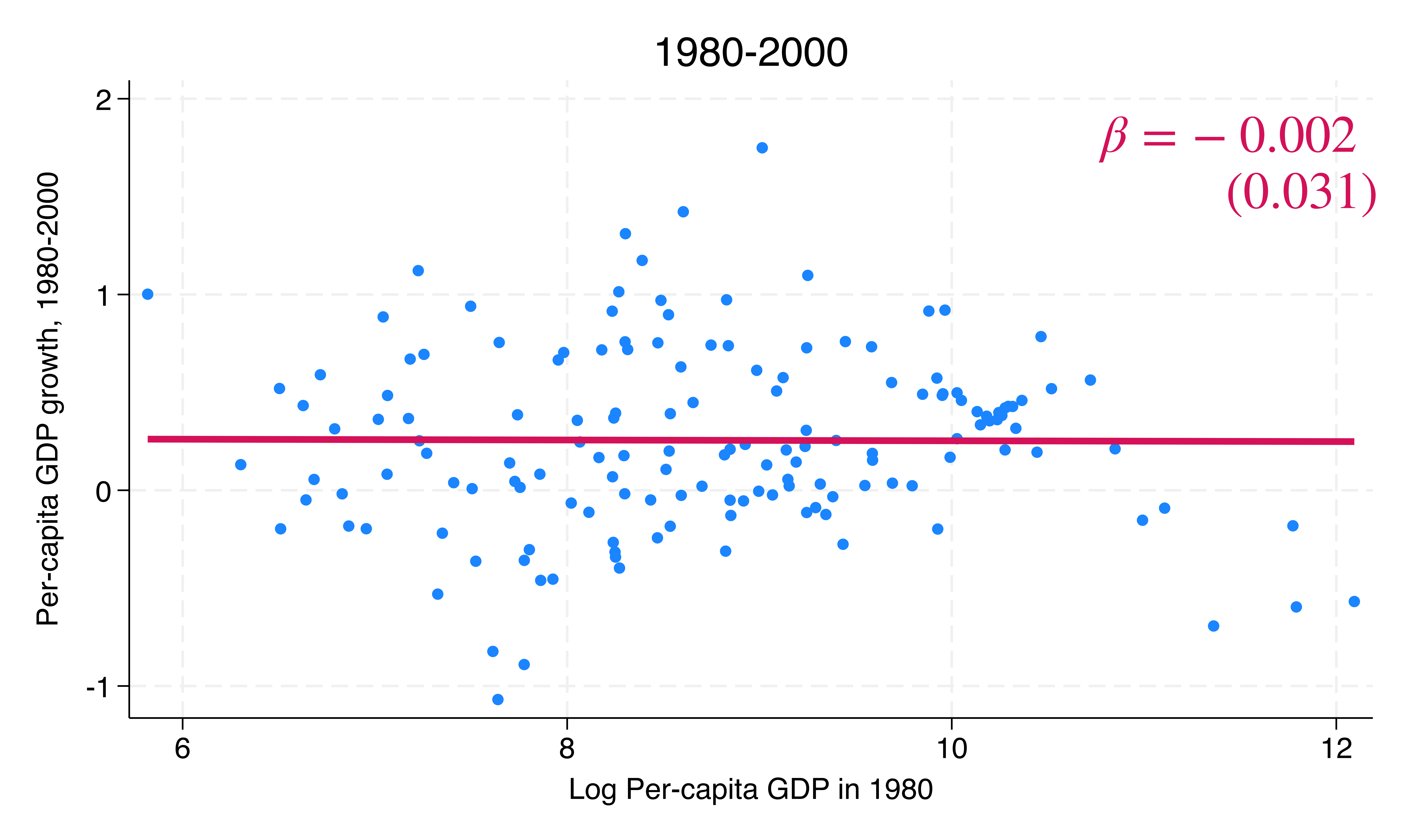


1960-2019
$\beta = -0.524$
(0.088)
Per-capita GDP growth, 1960-2019
Log Per-capita GDP in 1960
2.5
2
1.5
1
.5
8
8.5
9
9.5
10
10.5
ROU
CYP
IRL
PRT
TUR
ESP
ISL
FIN
LUX
NOR
AUT
GRC
BEL
ITA
FRA
DEU
NLD
GBR
CHE

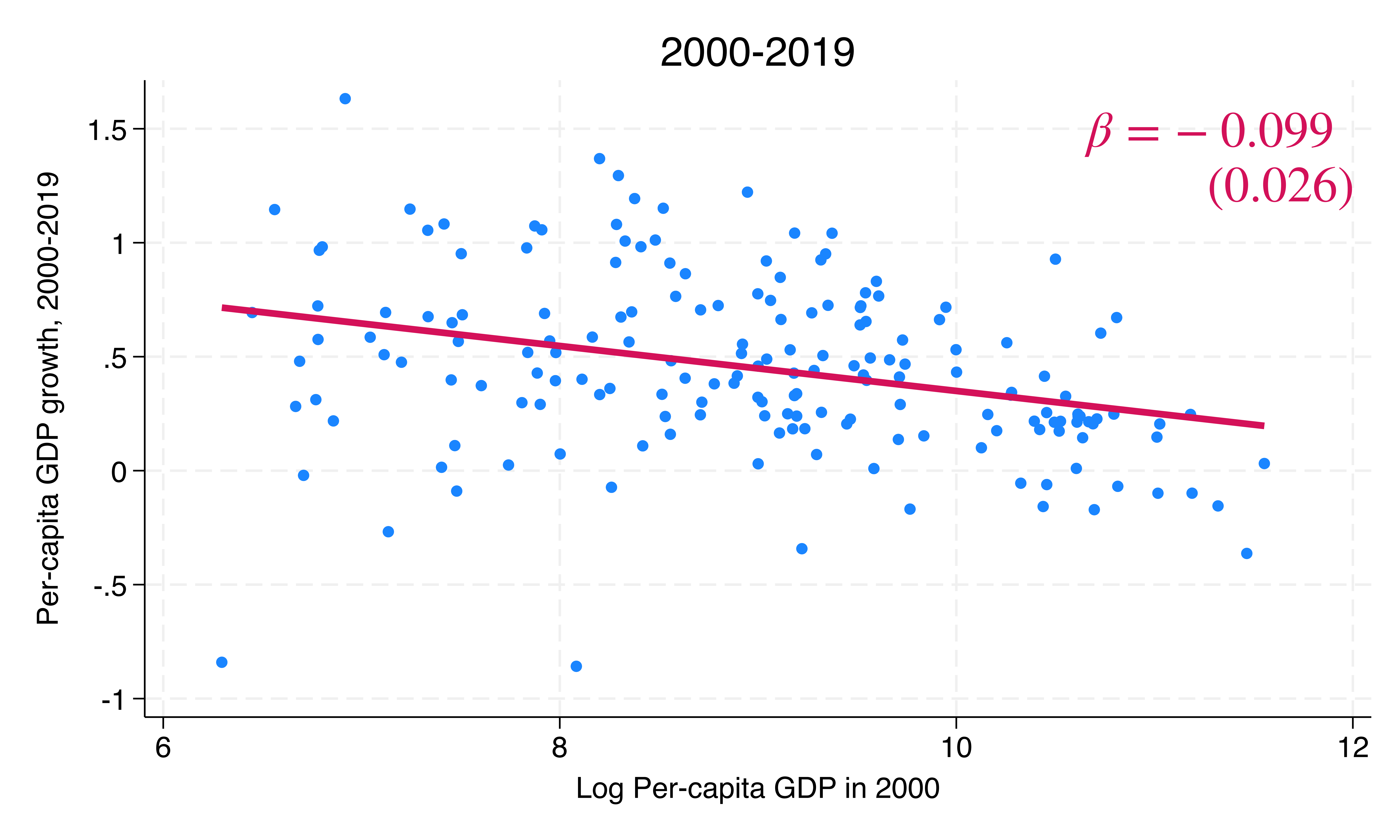
1960-1980


1960-1980
β = 0.075
(0.032)
Per-capita GDP growth, 1960-1980
Log Per-capita GDP in 1960
1.5
1
.5
0
-.5
6
7
8
9
10

1980-2000


1980-2000
$\beta = -0.002$
(0.031)
Per-capita GDP growth, 1980-2000
Log Per-capita GDP in 1980
2
1
0
-1
6
8
10
12

2000-2019



2000-2019
$\beta = -0.099$
(0.026)
1.5
1
.5
0
-.5
-1
Per-capita GDP growth, 2000-2019
6
8
10
12
Log Per-capita GDP in 2000

Interpretation

- Overall, there is no tendency of convergence
- We do see convergence
  1. if we focus on subsamples that look similar to each other
  2. if we only focus on recent periods
- Similar countries have similar $(A, s, \delta, \alpha, n)$, so the only difference is likely to be k_0
- Due to globalization, countries now have more similar fundamentals than before

Strength and Weakness of Solow Model

What Have We Learned?

Strength

- Provide a theory that determines the long-run level of k and y
  - based on primitive parameters: $(A, s, \delta, \alpha, n)$
- Its transition dynamics help us understand differences/changes in growth rates
  - The farther a country is below its steady state, the faster it will grow

Weakness

- Only provides a theory of k, not A
- Nothing to say about why countries differ in $(A, s, \delta, \alpha, n)$
- The model predicts no long-run growth