Ideas and Growth: Romer Model

EC502 Macroeconomics
Topic 3

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2025 Spring

What Sustains Long-run Growth?

- How do countries sustain long-run growth? Why is the US constantly growing at 2%?
- Solow model: capital accumulation cannot sustain growth in the long run
- Two reasons:
  1. Decreasing returns to scale in capital
     $\Rightarrow$ countries accumulate less and less capital as they grow
  2. Constant returns to scale in overall production
     $\Rightarrow$ more population do not lead to higher *per capita* income
- We will attack 2

Romer Model

Corn Farmer Example Again

- Farmers can use their labor (and capital) to produce corn
- … Or to invent new technologies for growing corn more productively

$$Y = F(A, L, K) = A^{\beta} L^{1-\alpha} K^{\alpha}$$

  - New tractors, fertilizer, irrigation systems, drought-resistant seed (ideas)
- What is the difference between objects and ideas?
  - Objects are rival
  - Ideas are non-rival
- Non-rivalry provides a natural foundation for increasing returns to scale
- Romer’s favorite example: oral rehydration therapy

Nonrivalry ⇒ Increasing Returns

$$Y = F(A, L, K) = A^{\beta} L^{1-\alpha} K^{\alpha}$$

- Replication argument: doubling (L, K) doubles Y:

$$F(A, 2L, 2K) = 2F(A, L, K)$$

- But doubling (A, L, K) more than doubles Y!

$$F(2A, 2L, 2K) > 2F(A, L, K)$$

- $F(A, L, K)$ is increasing returns to scale in (A, L, K)

Romer Model

- For simplicity, suppose there is no capital $(\alpha = 0)$

$$Y_t = A_t^{\beta} L_t$$

- Total population grows at rate n:

$$N_{t+1} = (1+n)N_t, \quad n > 0$$

- Fraction $1 - s^R$ of population engages in the production of goods:

$$L_t = (1 - s^R)N_t$$

- Fraction s^R of population engages in the production of ideas (R&D):

$$A_{t+1} = A_t + s^R N_t$$

Increasing Returns to Scale of Ideas

- The key assumption is, again, increasing returns to scale
- Per-capita output:

$$\frac{Y_t}{N_t} = (1 - s^R)A_t^{\beta}$$

  - Per-capita output is increasing in the **total** stock of knowledge
  - **Not** on knowledge per capita.
  - Reflects the fact that knowledge is non-rival

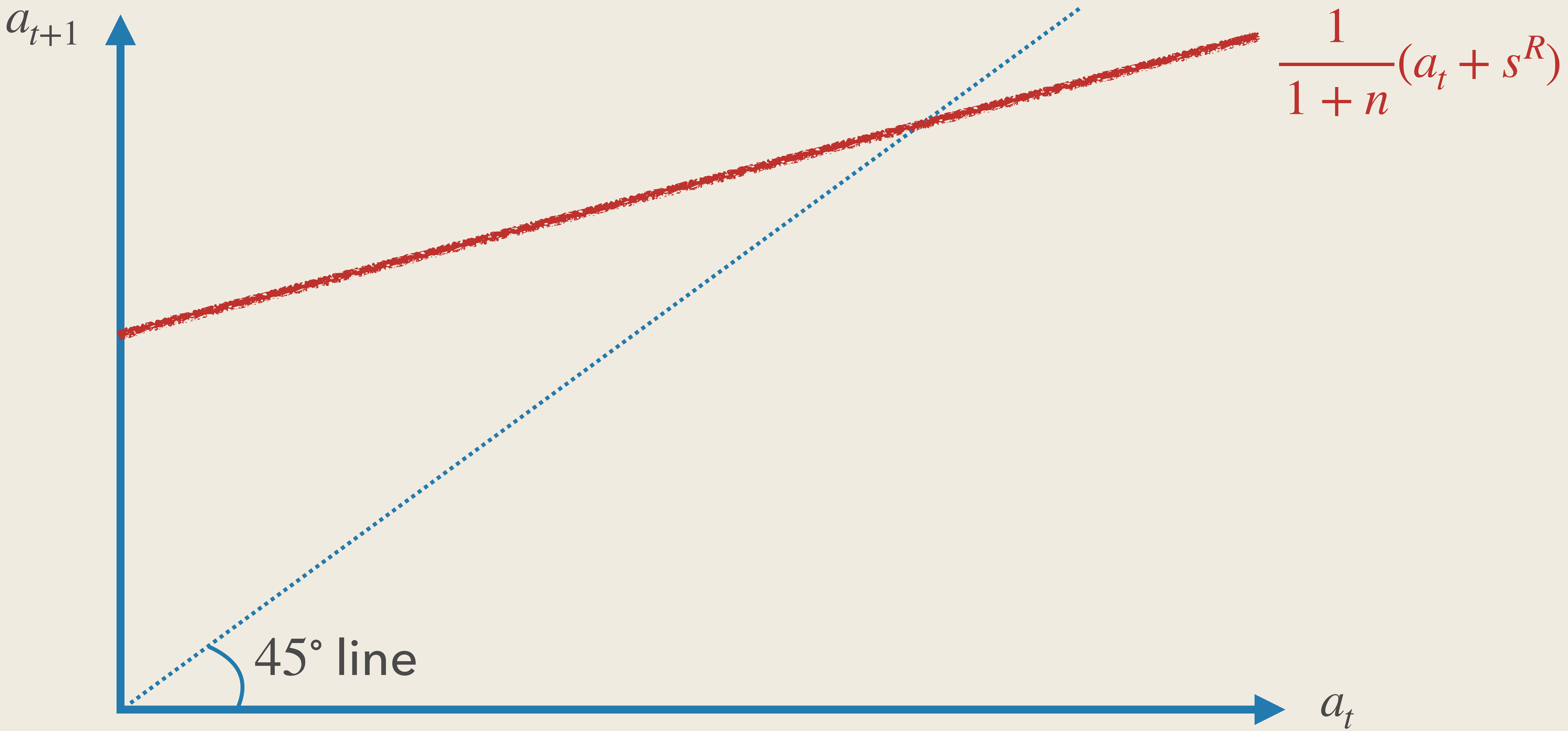
Knowledge Accumulation Process

- Define knowledge per capita: $a_t = A_t/N_t$
- Divide the knowledge accumulation equation by N_t to rewrite it as

$$a_{t+1} = \frac{1}{1+n}(a_t + s^R)$$

- Given a_0, the above equation determines $a_1, a_2, \ldots,$

Evolution of Knowledge Stock



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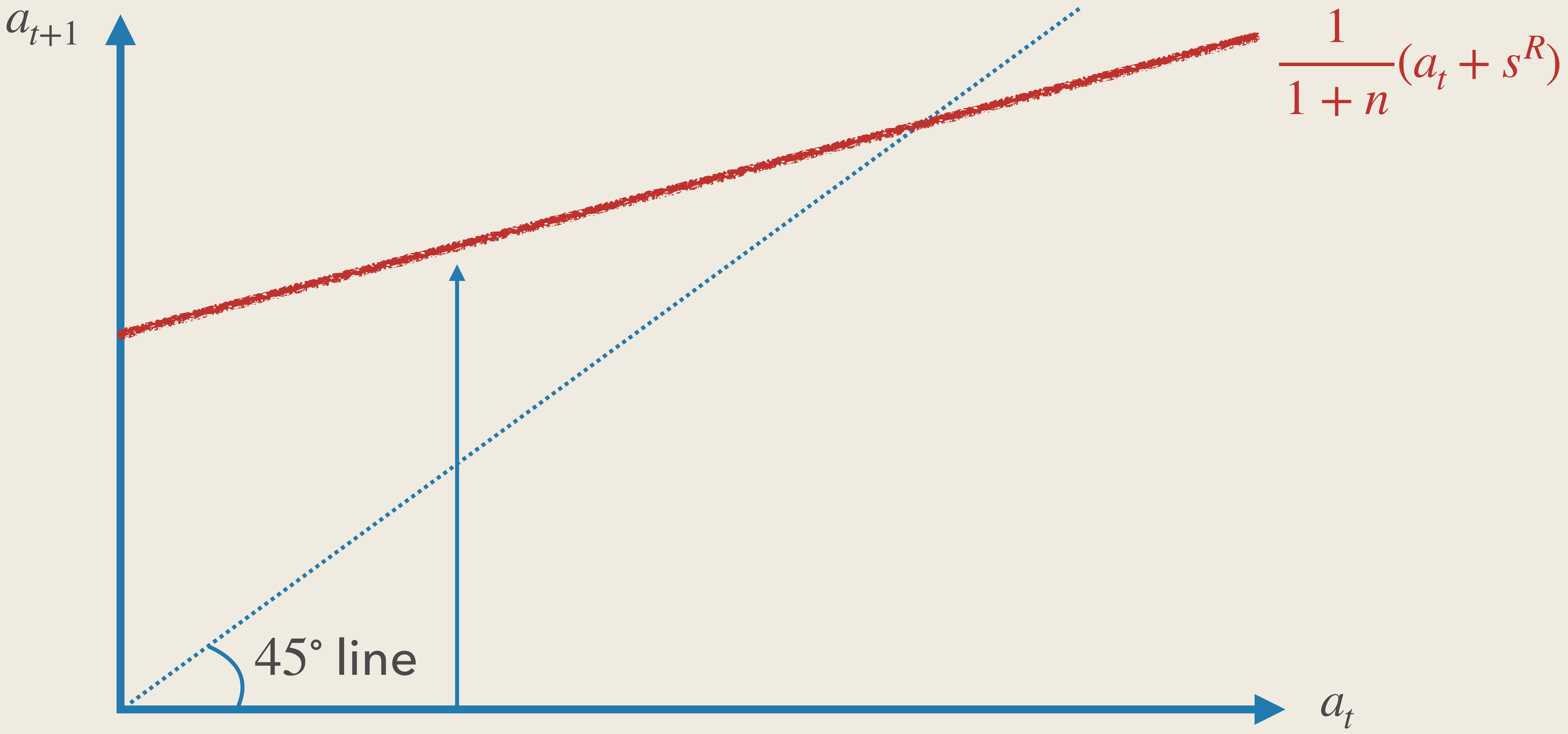

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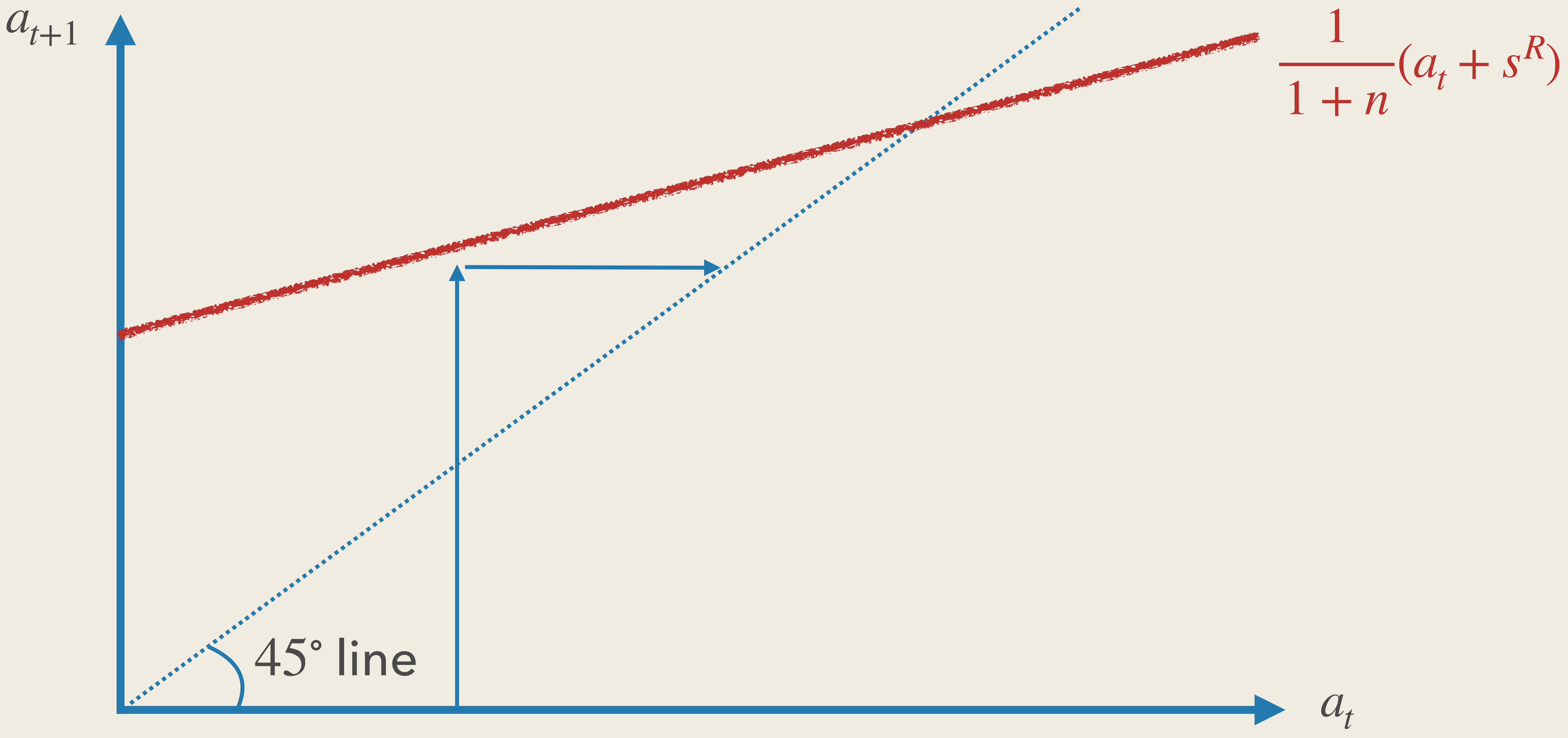

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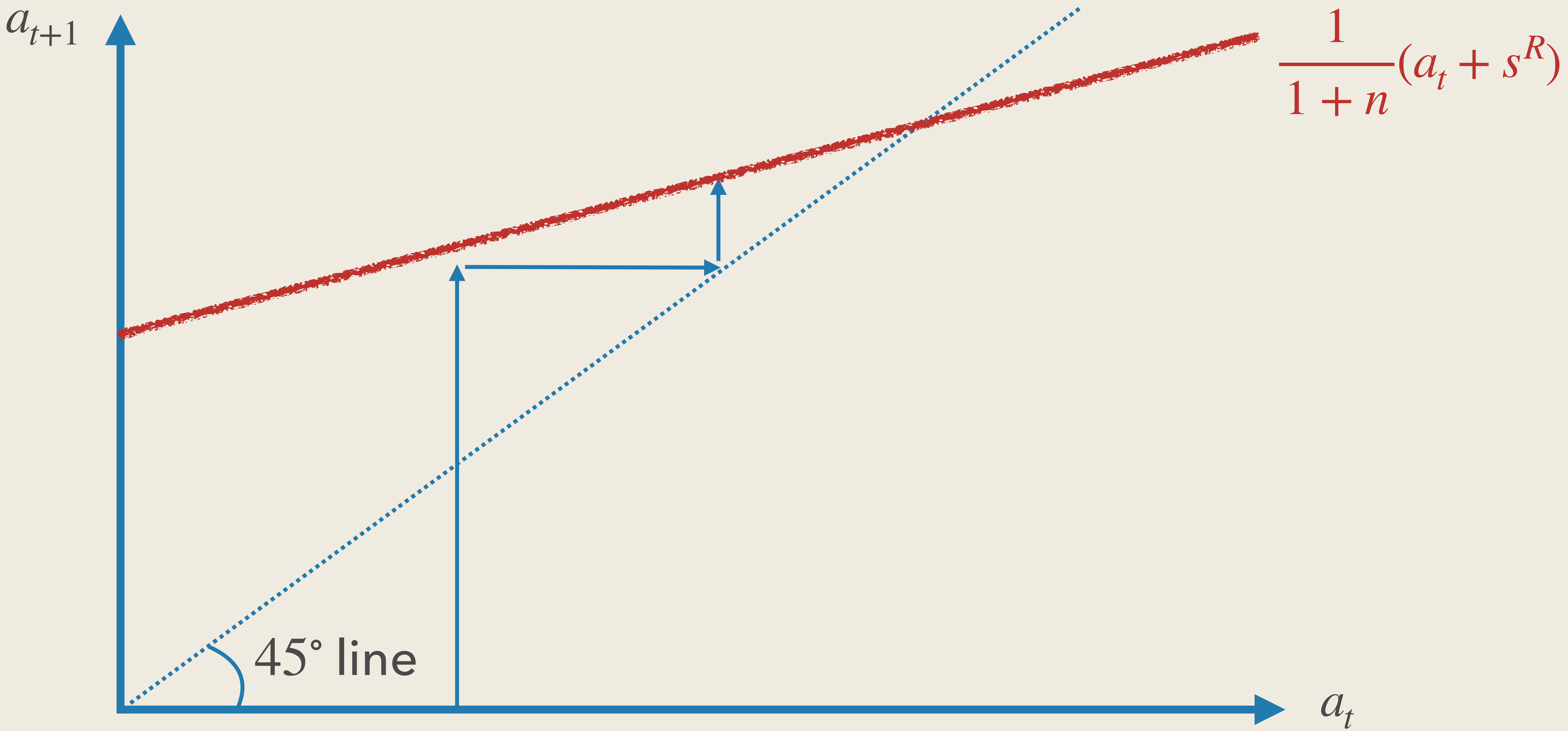

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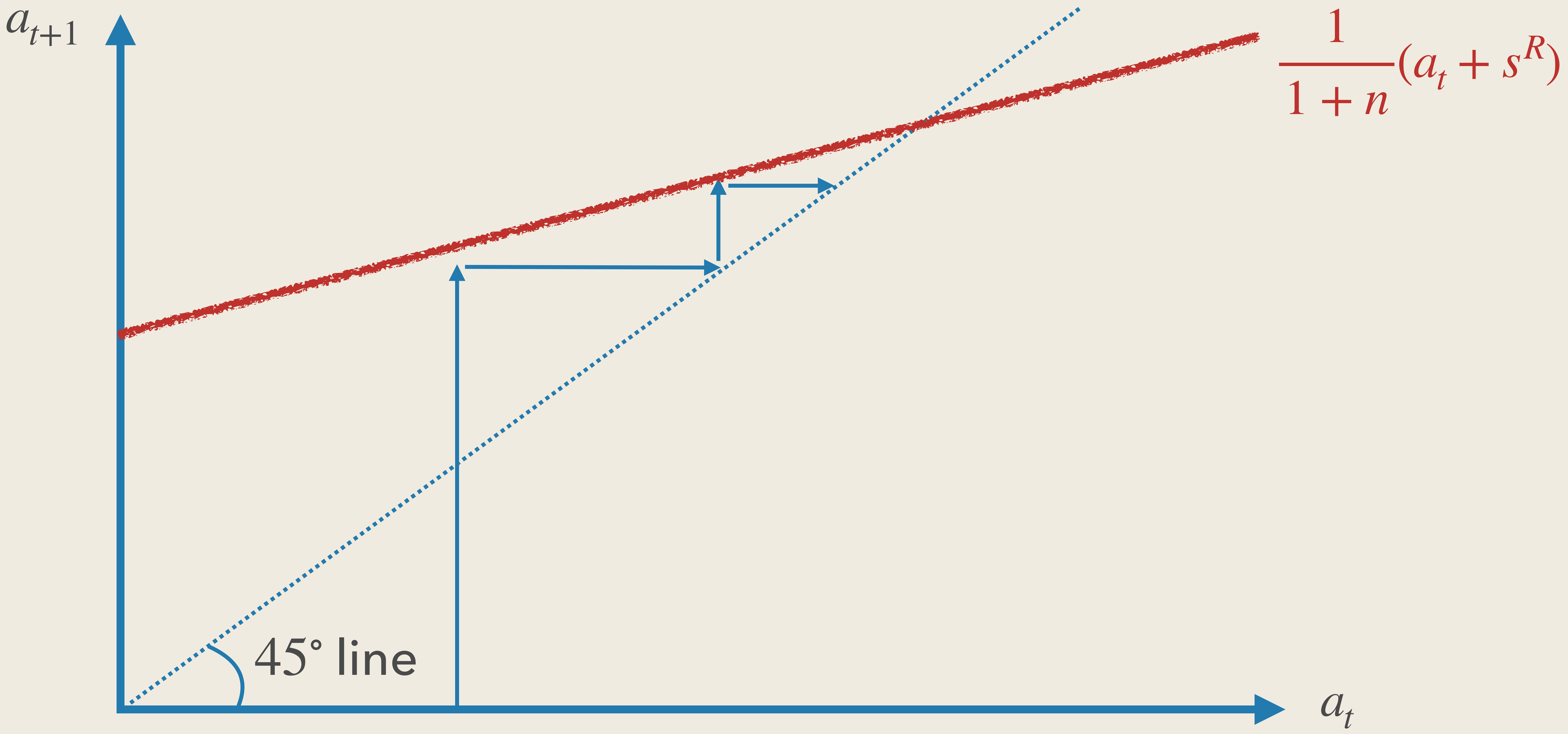

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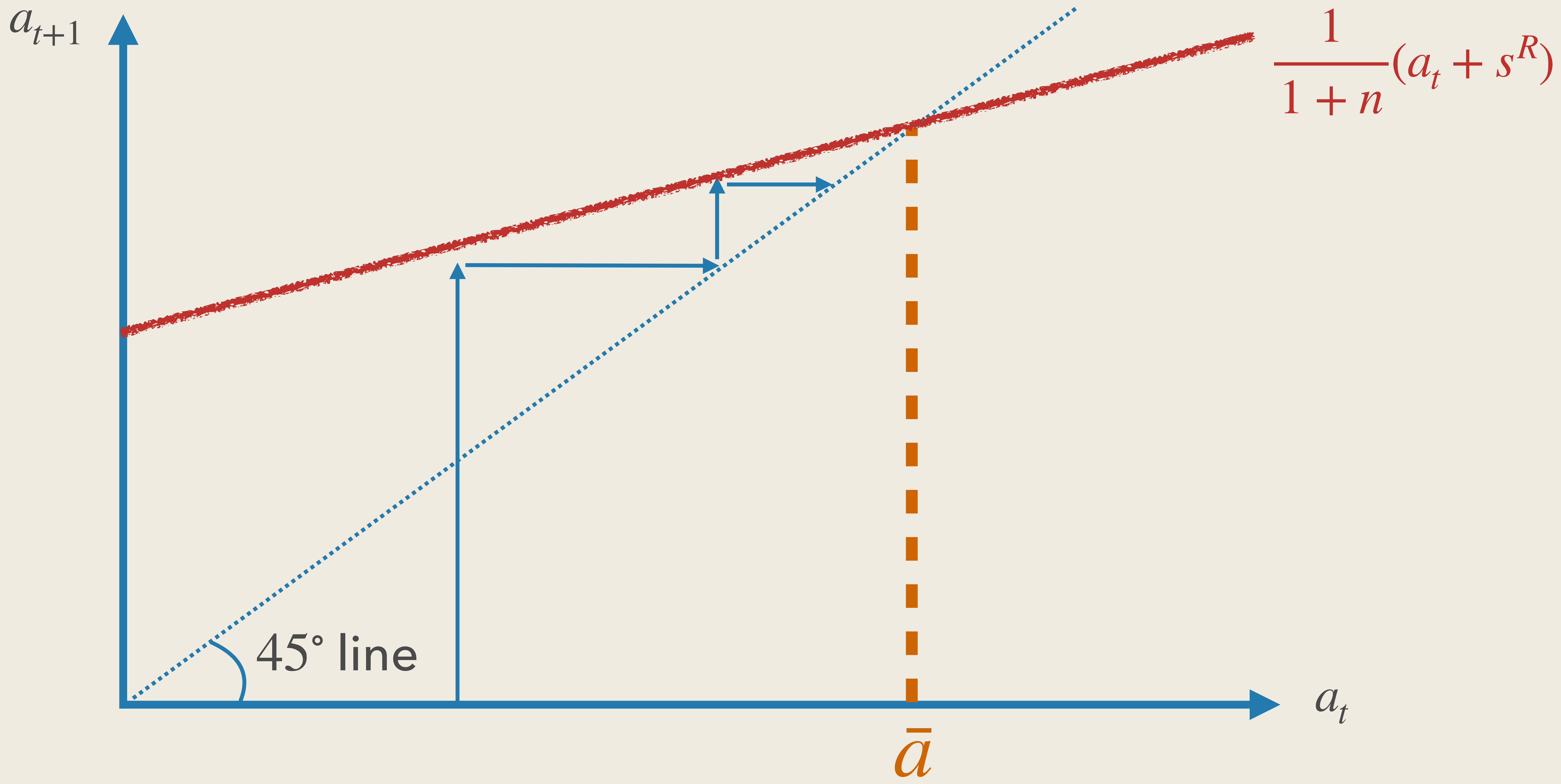

Long-run Growth in Knowledge

- In the long-run (**steady state**), the knowledge per capita converges to $\bar{a}$ that satisfies

$$\bar{a} = \frac{1}{1+n}(\bar{a} + s^R)$$

- Solving for $\bar{a}$ gives $\bar{a} = s^R/n$.

- More importantly, $\bar{a}$ constant $\Rightarrow A_t = \bar{a}N_t \Rightarrow A_t$ keeps growing at the speed N_t grows

$$1 + g_A = \frac{A_{t+1}}{A_t} = \frac{N_{t+1}}{N_t} = 1 + n$$

- Growth rate of knowledge = growth rate of researchers = population growth

Long-run Growth in GDP per capita!

- Recall per-capita output is $Y_t/N_t = (1 - s^R)A_t^\beta$
- The growth rate of per-capita output is

$$1 + g_{Y/N} \equiv \log(Y_{t+1}/N_{t+1}) - \log(Y_t/N_t) = \log A_{t+1}^\beta - \log A_t^\beta = \beta \log(1+n)$$

- When n is small,

$$g_{Y/N} \approx \beta n$$

- Per-capita GDP growth = importance of knowledge (β) $\times$ population growth (n)
- A country sustains long-run growth in GDP per capita!

Combining Romer and Solow Model

Solow + Romer

- Now we put back capital

$$Y_t = A_t^{\beta} L_t^{1-\alpha} K_t^{\alpha}$$

$$N_{t+1} = (1+n)N_t$$

$$L_t = (1 - s^R)N_t$$

$$A_{t+1} = A_t + s^R N_t$$

$$K_{t+1} = K_t(1-\delta) + sY_t$$

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Convenient Normalization

- Educated trick: we normalize all variables with $A_t^{\frac{\beta}{1-\alpha}} N_t$
  - This makes all variables stationary, as we will see
- Define $y_t = Y_t / (A_t^{\frac{\beta}{1-\alpha}} N_t)$ and $k_t = K_t / (A_t^{\frac{\beta}{1-\alpha}} N_t)$. Then

$$y_t = (1 - s_R)^{1-\alpha} k_t^{\alpha} \tag{1}$$

$$k_{t+1} = \frac{1}{(1 + g_{At})^{\frac{\beta}{1-\alpha}}} \frac{1}{(1+n)} \left[k_t (1 - \delta) + s (1 - s_R)^{1-\alpha} k_t^{\alpha} \right] \tag{2}$$

$$A_{t+1}/N_{t+1} = \frac{1}{1+n} \left[A_t / N_t + s^R \right] \tag{3}$$

where $1 + g_{At} = A_{t+1}/A_t$

Derivations of (1)

- Production function:

$$Y_t = A_t^{\beta} L_t^{1-\alpha} K_t^{\alpha}$$

- Divide both sides by $A_t^{\frac{\beta}{1-\alpha}} N_t$:

$$\frac{Y_t}{A_t^{\frac{\beta}{1-\alpha}} N_t} = \frac{A_t^{\beta}}{A_t^{\frac{\beta}{1-\alpha}}} \left(\frac{L_t}{N_t}\right)^{1-\alpha} \left(\frac{K_t}{N_t}\right)^{\alpha}$$

$$= \frac{1}{A_t^{\frac{\beta\alpha}{1-\alpha}}} \left(1 - s^R\right)^{1-\alpha} \left(\frac{K_t}{N_t}\right)^{\alpha}$$

$$= \left(1 - s^R\right)^{1-\alpha} \left(\frac{K_t}{A_t^{\frac{\beta}{1-\alpha}} N_t}\right)^{\alpha}$$

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$$= \left(1 - s^R\right)^{1-\alpha} \left(\frac{K_t}{A_t^{\frac{\beta}{1-\alpha}} N_t}\right)^{\alpha}$$

y_t

k_t

Derivation of (2)

$$K_{t+1} = K_t(1-\delta) + sY_t$$

- Divide both sides by $A_t^{\frac{\beta}{1-\alpha}} N_t$:

$$\frac{K_{t+1}}{A_t^{\frac{\beta}{1-\alpha}} N_t} = k_t(1-\delta) + sy_t$$

- Multiply and divide the left-hand side by $A_{t+1}^{\frac{\beta}{1-\alpha}} N_{t+1}$

$$\frac{K_{t+1}}{A_{t+1}^{\frac{\beta}{1-\alpha}} N_{t+1}} \frac{A_{t+1}^{\frac{\beta}{1-\alpha}}}{A_t^{\frac{\beta}{1-\alpha}}} \frac{N_{t+1}}{N_t} = k_t(1-\delta) + sy_t$$

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k_{t+1}

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$$\frac{K_{t+1}}{A_{t+1}^{\frac{\beta}{1-\alpha}} N_{t+1}} \frac{A_{t+1}^{\frac{\beta}{1-\alpha}}}{A_t^{\frac{\beta}{1-\alpha}}} \frac{N_{t+1}}{N_t} = k_t(1-\delta) + sy_t$$

k_{t+1} $\quad (1+g_{At})^{\frac{\beta}{1-\alpha}} \quad$ $1+n$

Solow + Romer in the Long-run

- The previous equations dictate the dynamics of $a_t = A_t/N_t$ and k_t
- Let us focus on the long-run
- In the long-run, as we have seen already, (3) implies $a_t = A_t/N_t$ is a constant, and

$$g_{At} = g_A = n$$

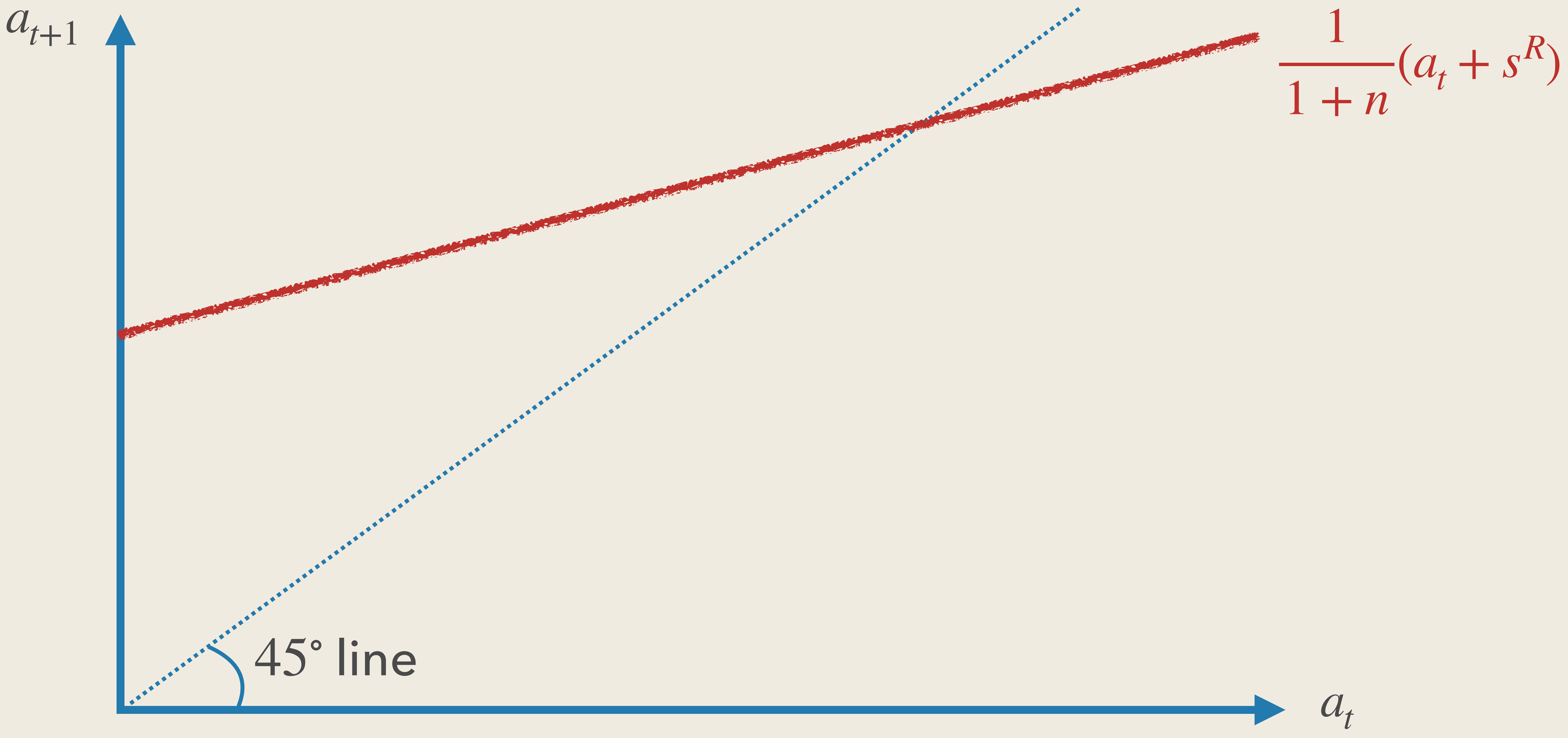
- Putting $g_A = n$ into (2), we now obtain a nearly identical equation as in Solow model:

$$k_{t+1} = \frac{1}{(1+n)^{1+\beta/(1-\alpha)}}\left[k_t(1-\delta) + s(1-s^R)^{1-\alpha}k_t^\alpha\right]$$

- In the long-run with $k_t = k$,

$$k = \left(\frac{s(1-s^R)^{1-\alpha}}{(1+n)^{1+\beta/(1-\alpha)} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}, \quad y = (1-s^R)^{1-\alpha}k^\alpha$$

Evolution of Knowledge Stock



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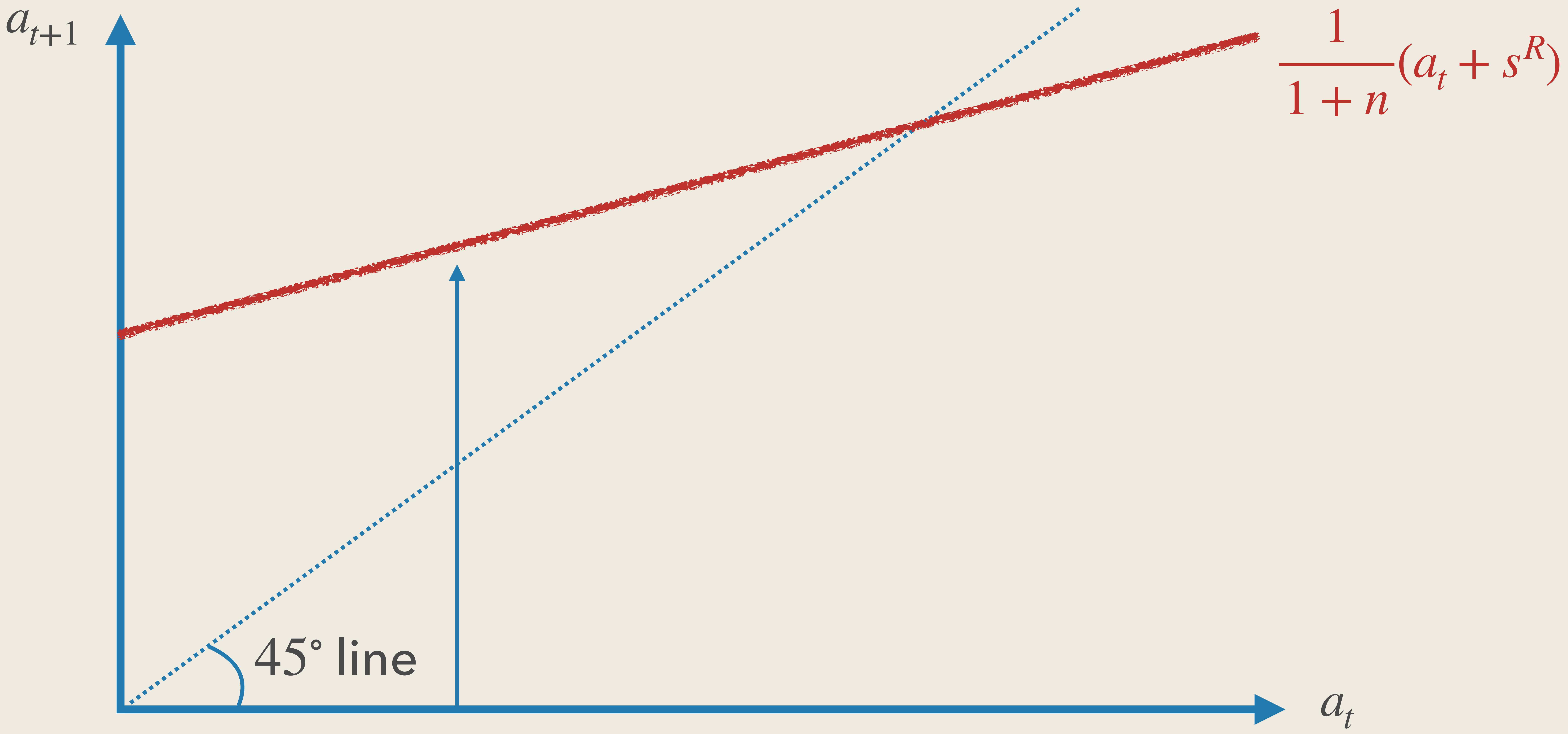

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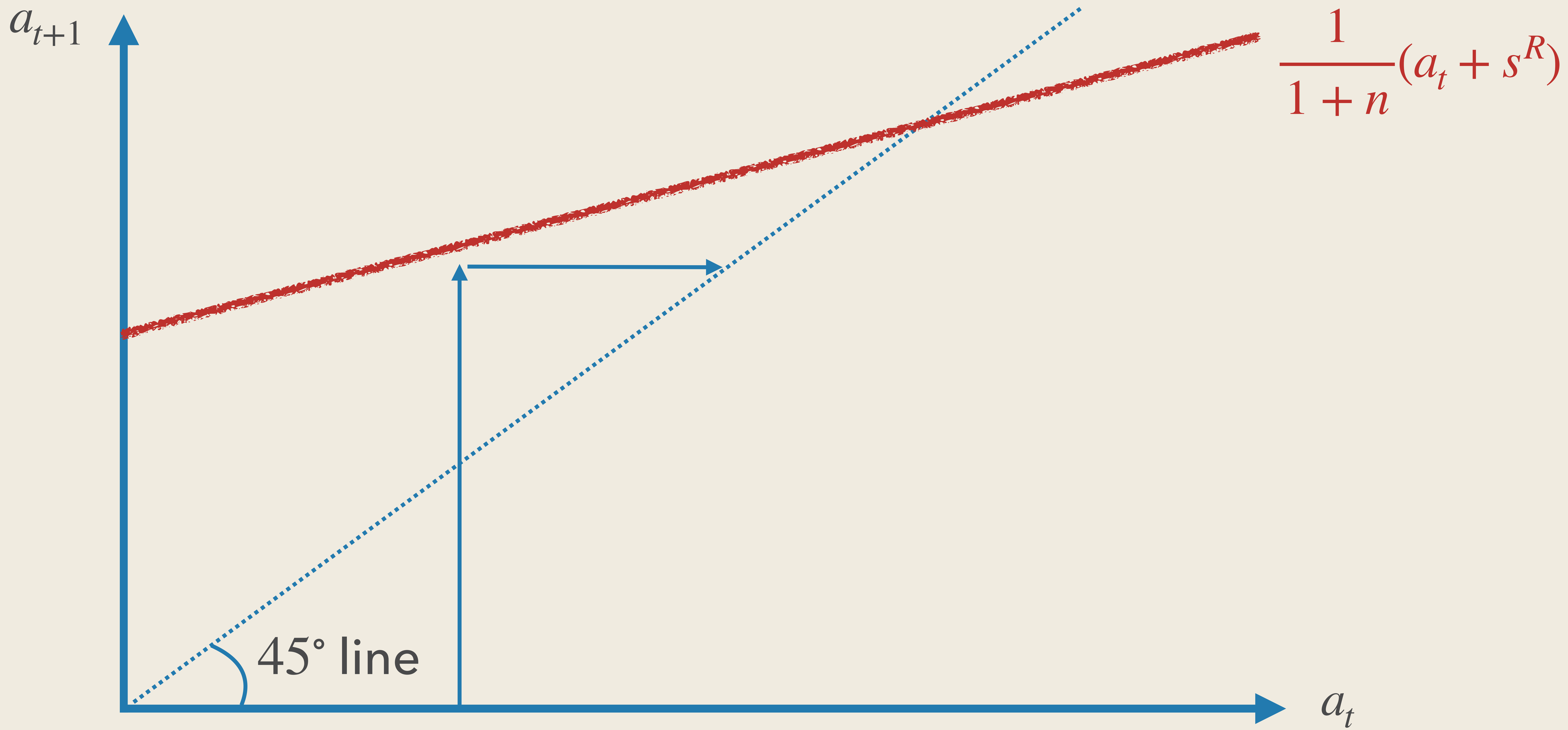

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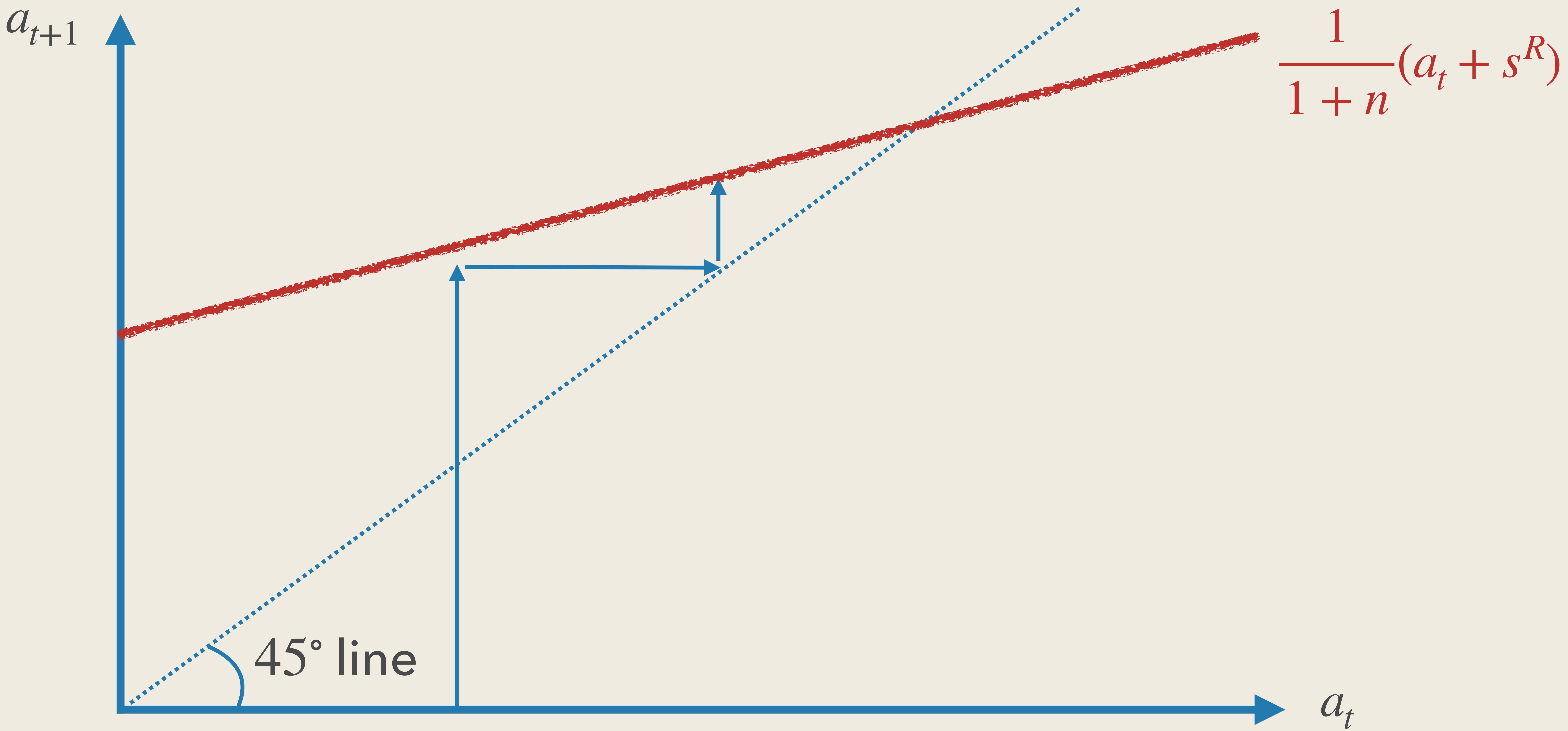

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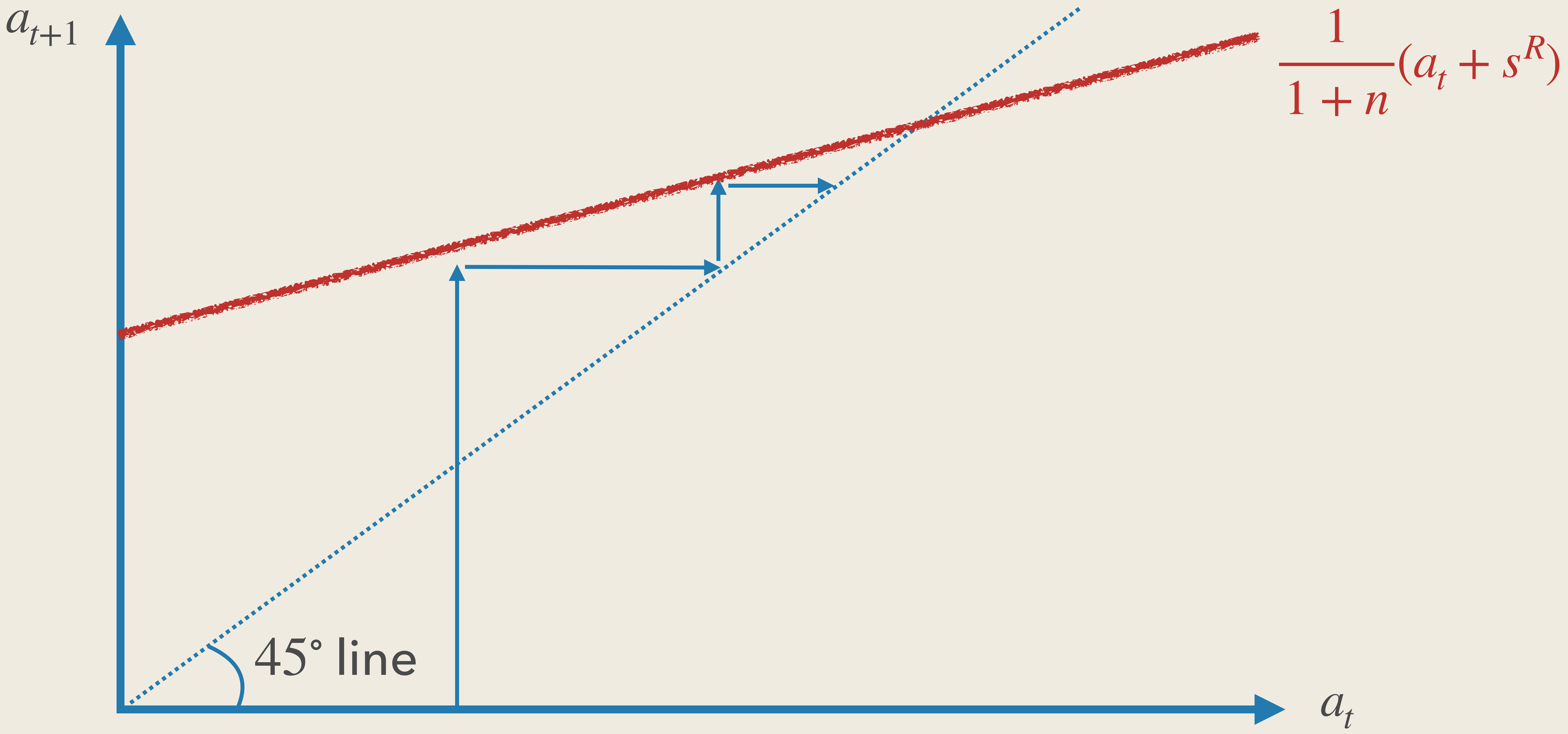

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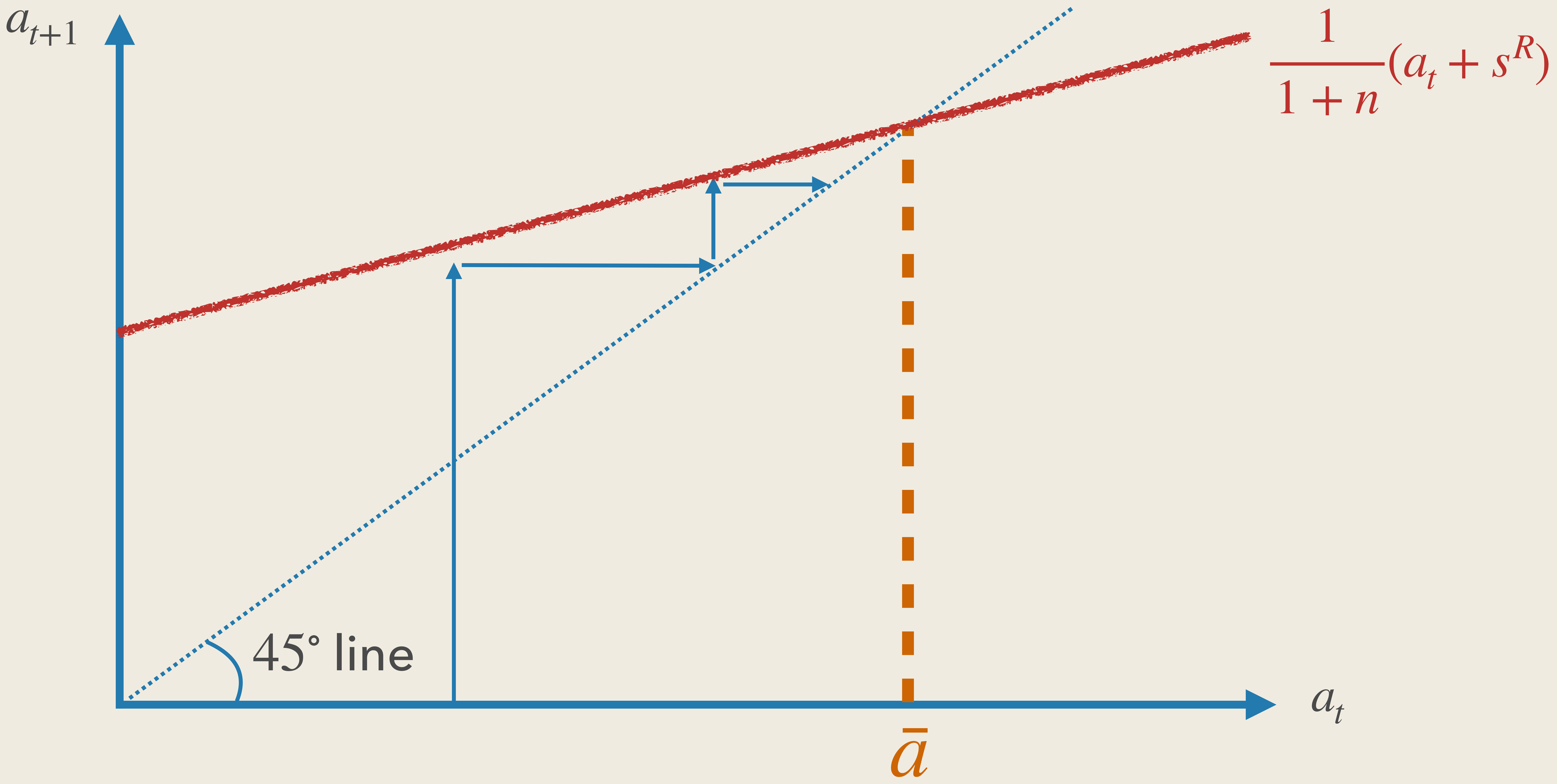
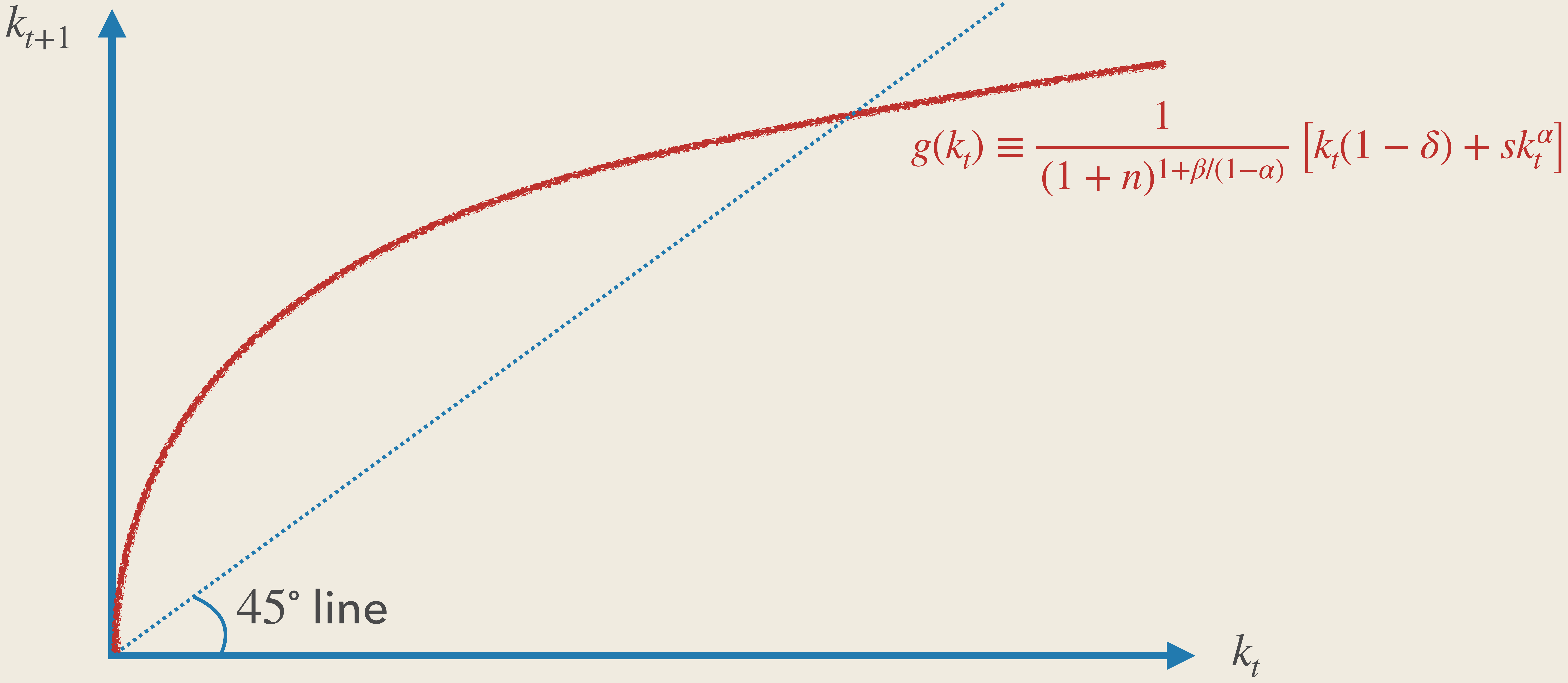


Evolution of Capital Stock



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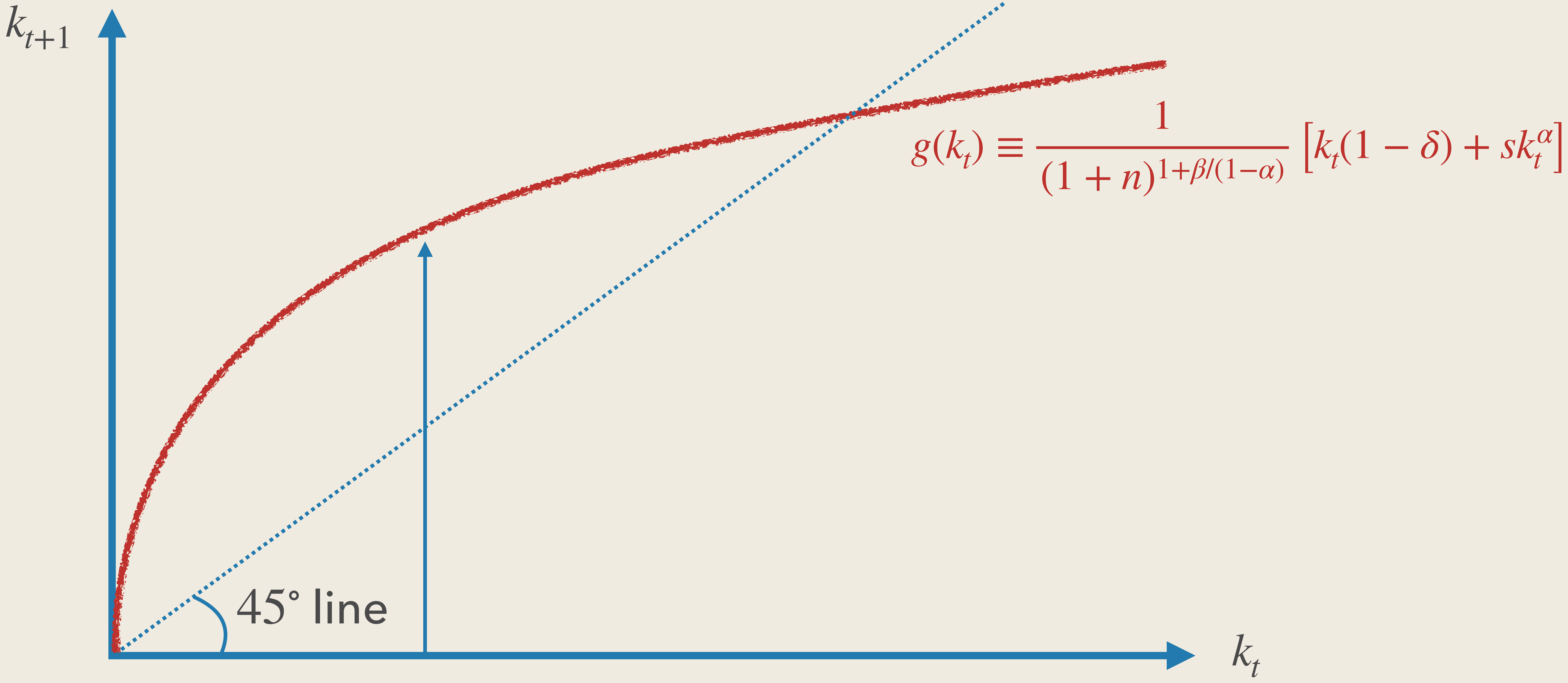

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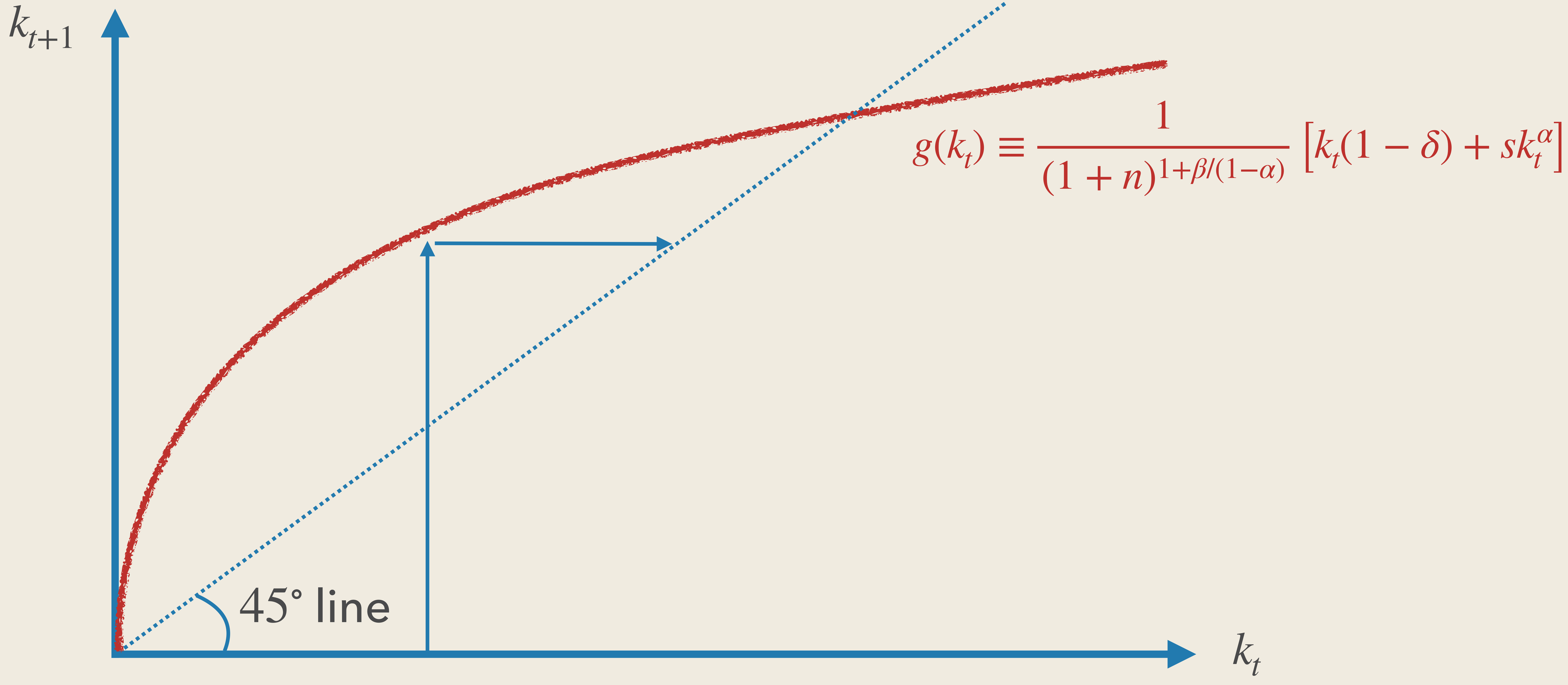

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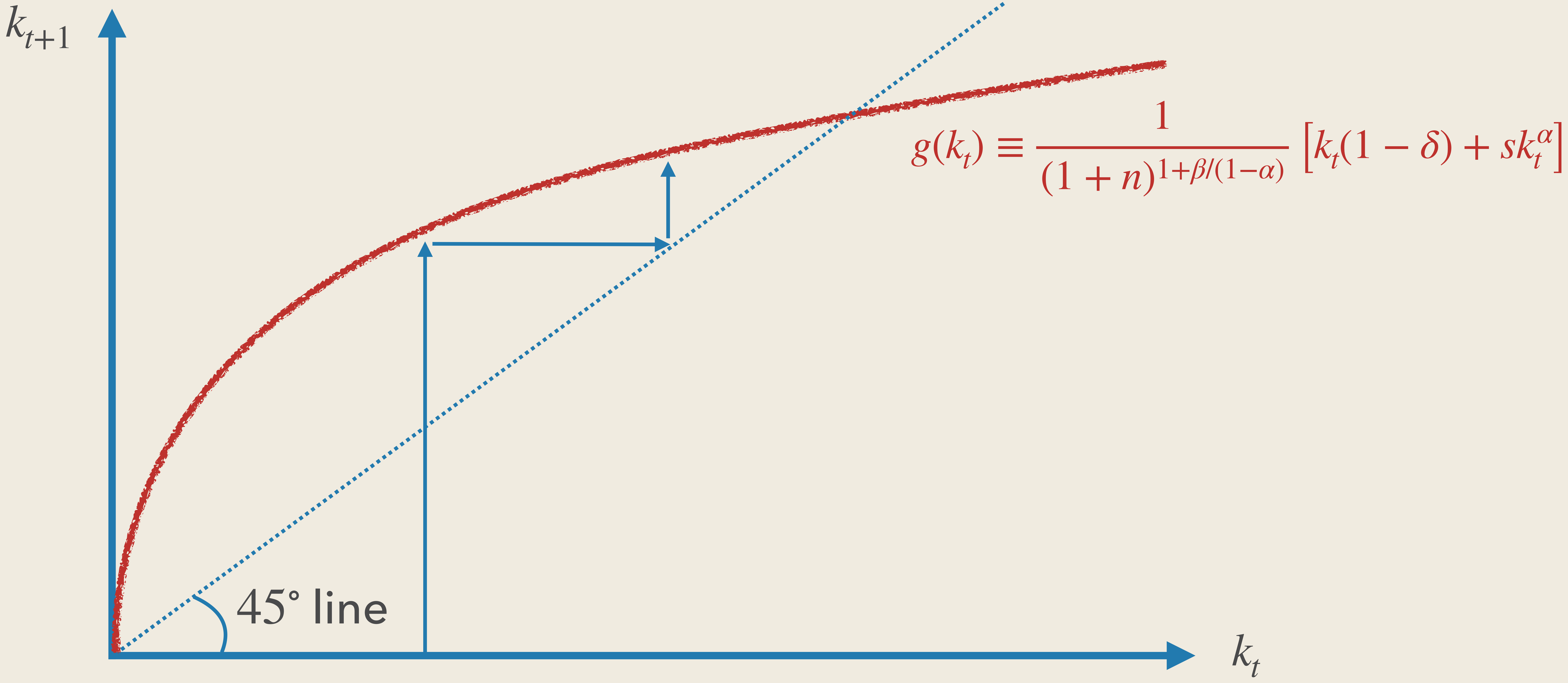

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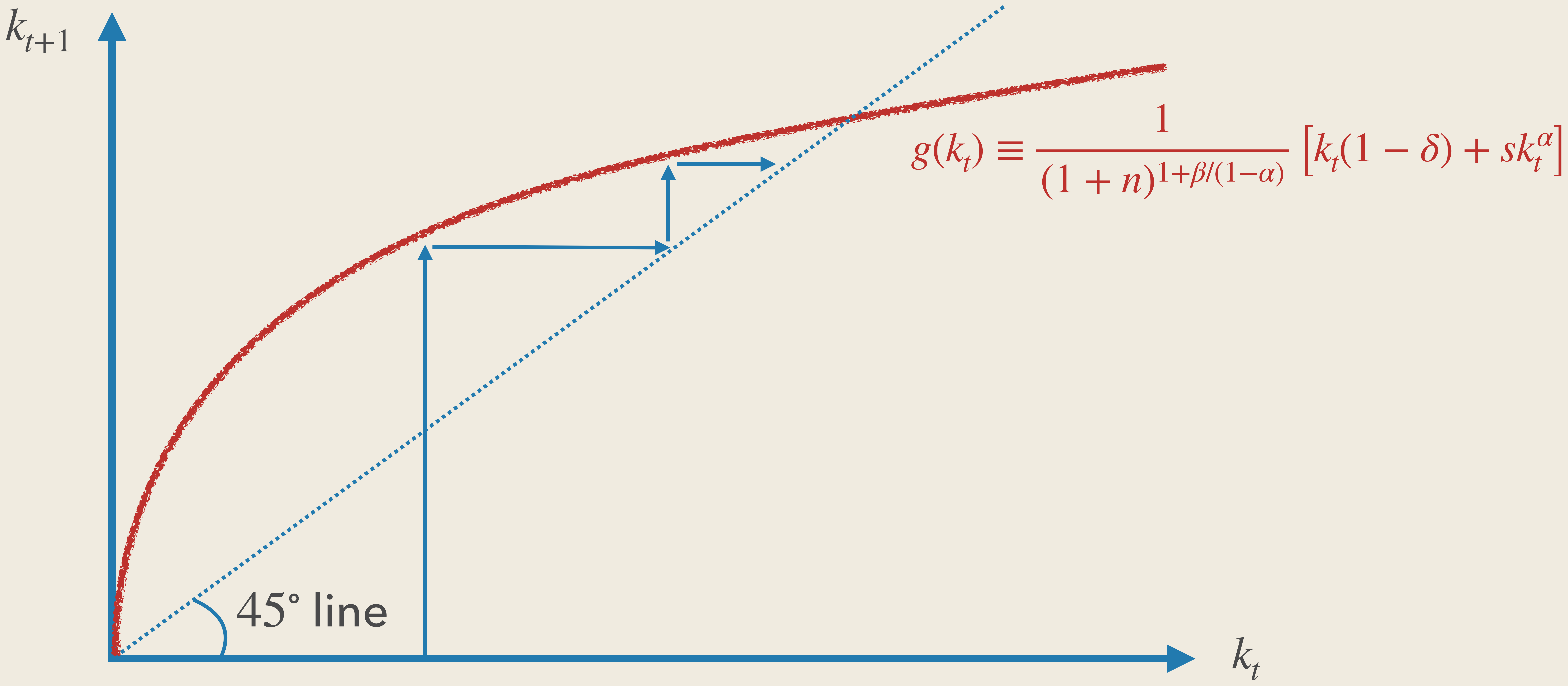

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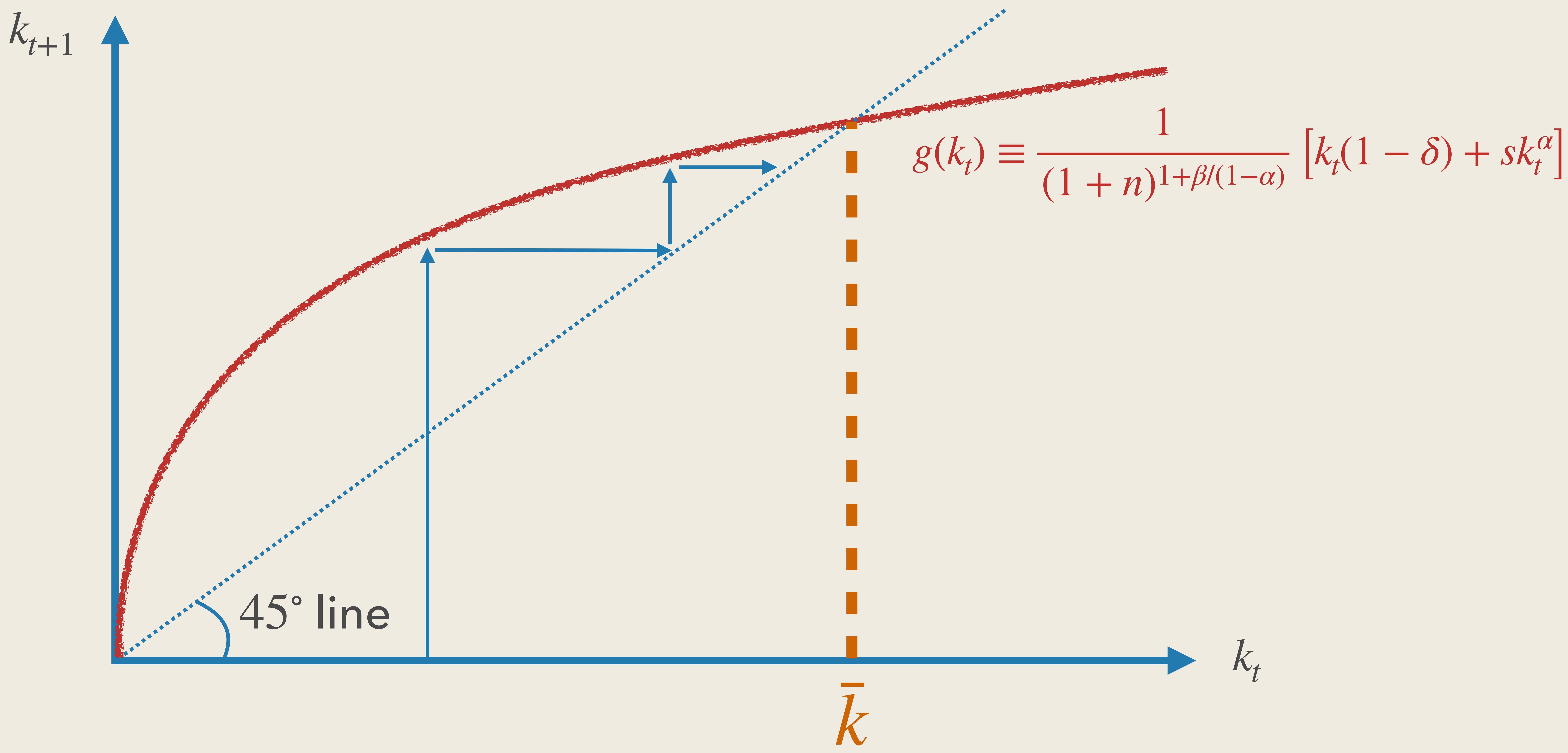


Long-run Growth

- What is the long-run growth in this economy?
- Does the fact that y is a constant mean per-capita income is also a constant?
  - No, recall per capita income is $Y_t/N_t = A_t^{\frac{\beta}{1-\alpha}} y_t$
    … which grows at a rate $g_{Y/N} \approx \frac{\beta}{1-\alpha} n$ (when n small)
- The economy grows faster than the previous model (which was βn). Why?

$$Y_t/N_t = (1 - s_R)^{1-\alpha} A_t^{\beta} (K_t/N_t)^{\alpha}$$

  - A_t grow at rate $g_A = n$ and contribute to GDP growth by βg_A
  - K_t/N_t grow at rate $\frac{\beta}{1-\alpha} g_A$ and contribute to GDP growth by $\frac{\alpha\beta}{1-\alpha} g_A$
  - Technology growth leads to capital accumulation and even faster growth

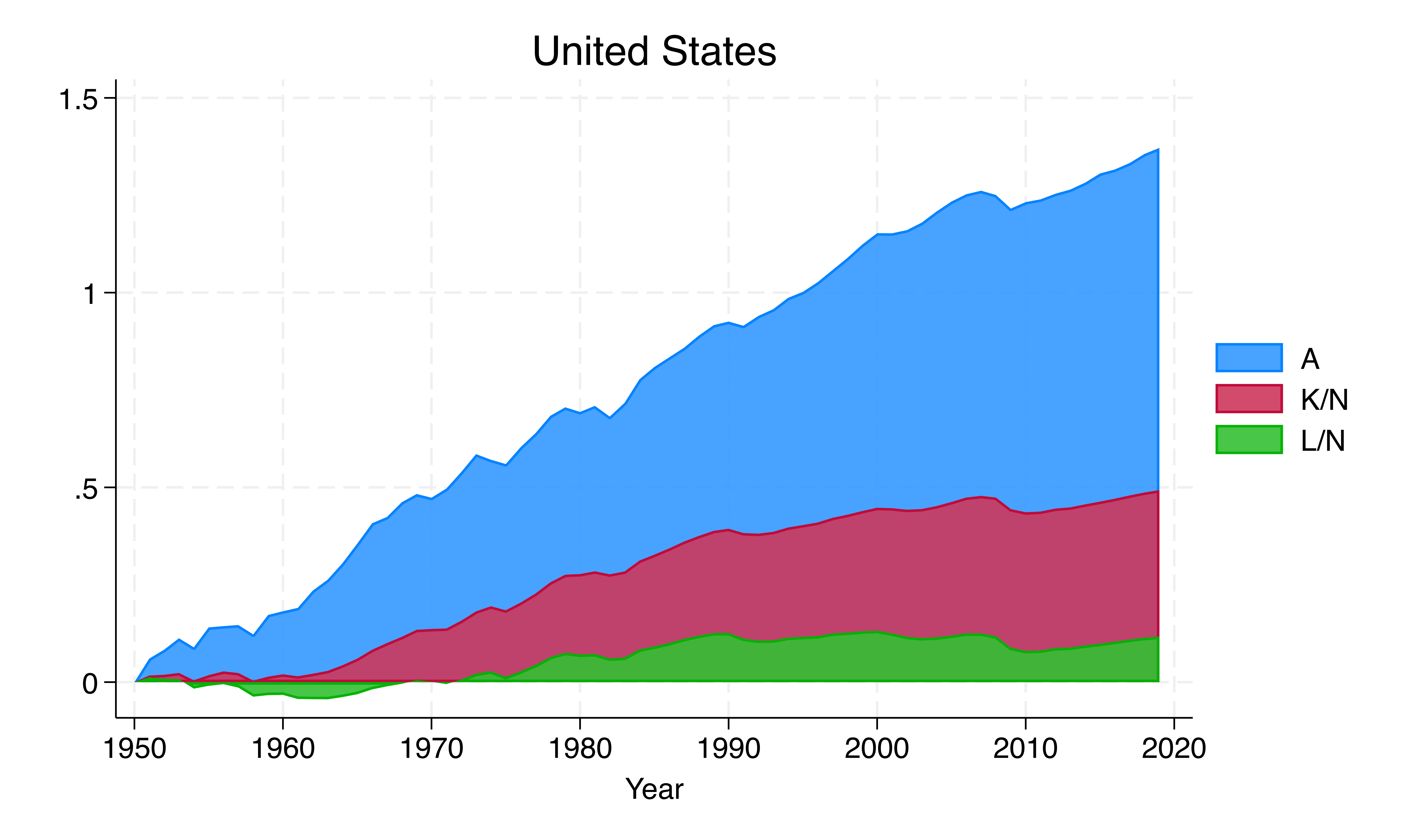
Growth Accounting Revisided

- In this model,

$$g_{Y/N} = \underbrace{\beta g_A}_{\text{growth due to A}} + \underbrace{\frac{\alpha}{1-\alpha}\beta g_A}_{\text{growth due to K}}$$

- If $\alpha = 1/3$, A should be twice as important as capital in growth accounting
- Let us go back to the data and test it

Validating Solow + Romer Model



Putting a Number

$$g_{Y/N} = \frac{\beta}{1-\alpha} n$$

- Per-capita GDP in the US has been growing at 2% every year, $g_{Y/N} = 0.02$
- The US population has been growing roughly at 1%, $n = 0.01$
- Labor share implies $\alpha = 1/3$
- Jointly, this implies $\beta \approx 1.33$

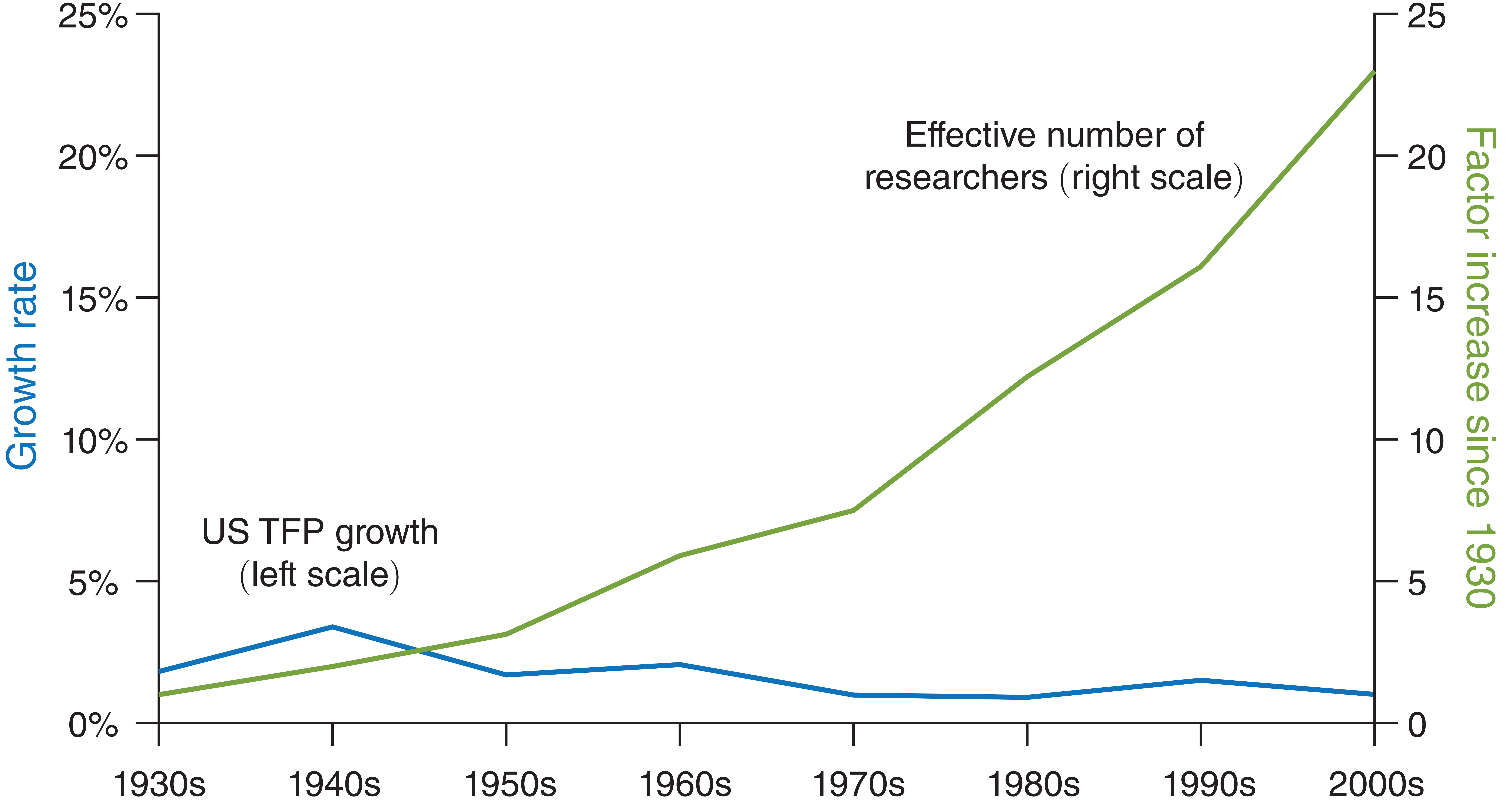
1. Are Ideas Getting Harder to Find? – Bloom, Jones, Van Reenen, Webb (2020)

Are Ideas Getting Harder to Find?

Economic growth = (research productivity) × (No. of researchers)

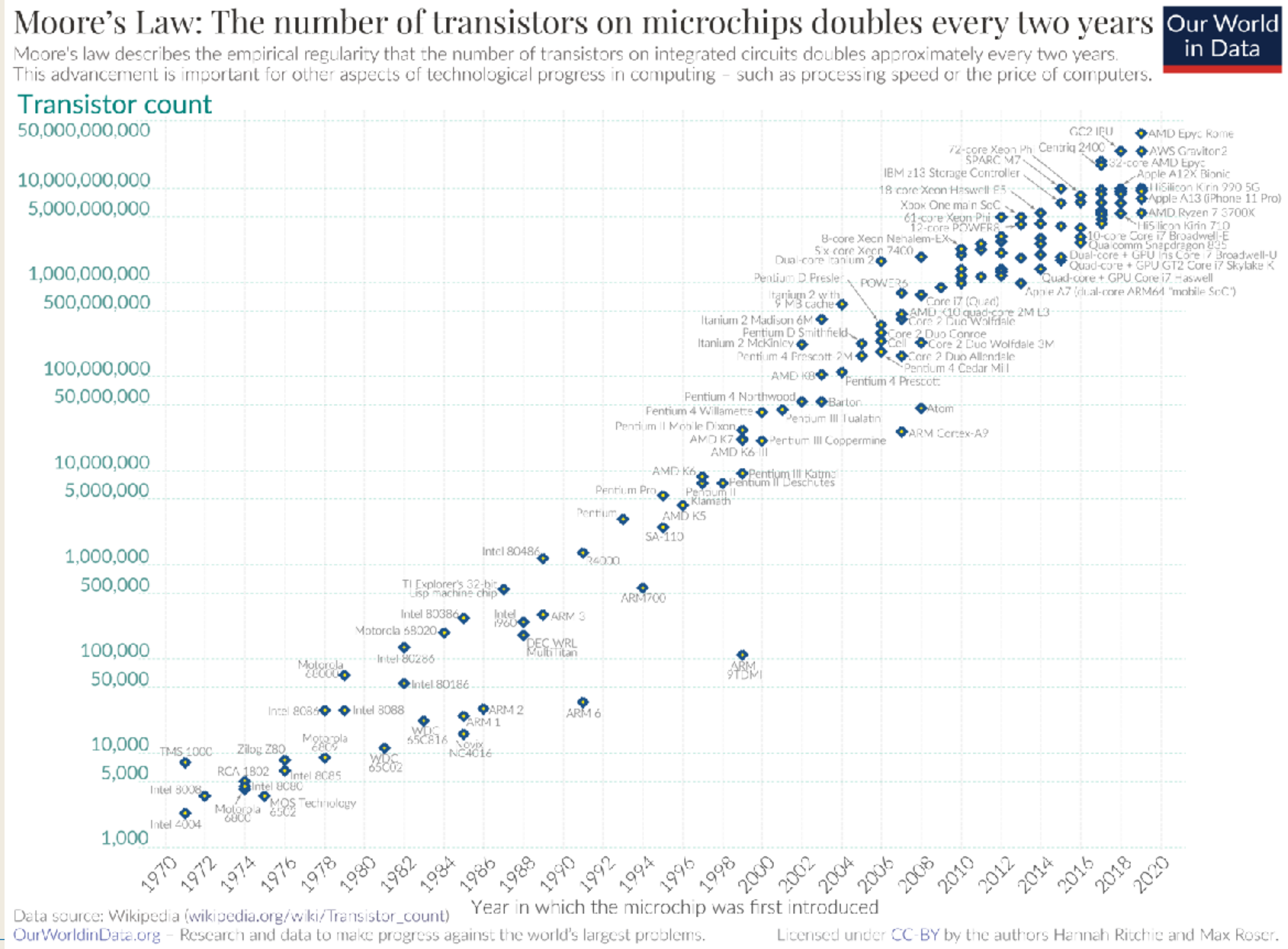
- Romer model strongly ties GDP growth to the growth of researchers
- In order to sustain GDP growth, we need more and more researchers
- This means ideas get harder and harder to find as a country grows

Researchers and TFP Growth: Aggregate Data



Source: Bloom, Jones, Van Reenen, & Webb (2020)

Moore's Law



Moore's Law: The number of transistors on microchips doubles every two years
Our World in Data
Moore's law describes the empirical regularity that the number of transistors on integrated circuits doubles approximately every two years.
This advancement is important for other aspects of technological progress in computing – such as processing speed or the price of computers.
Transistor count
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AMD Epyc Rome
AWS Graviton2
32-core AMD Epyc
Apple A12X Bionic
HiSilicon Kirin 990 5G
Apple A13 (iPhone 11 Pro)
AMD Ryzen 7 3700X
HiSilicon Kirin 710
10-core Core i7 Broadwell-E
Qualcomm Snapdragon 835
Dual-core + GPU Iris Core i7 Broadwell-U
Quad-core + GPU GT2 Core i7 Skylake K
Quad-core + GPU Core i7 Haswell
Apple A7 (dual-core ARM64 "mobile SoC")
GC2 IPU
Centriq 2400
72-core Xeon Phi
SPARC M7
IBM z13 Storage Controller
18-core Xeon Haswell-E5
Xbox One main SoC
61-core Xeon Phi
12-core POWER8
8-core Xeon Nehalem-EX
Six-core Xeon 7400
Dual-core Itanium 2
Pentium D Presler
POWER6
Itanium 2 with 9 MB cache
Core i7 (Quad)
AMD K10 quad-core 2M L3
Core 2 Duo Wolfdale
Itanium 2 Madison 6M
Pentium D Smithfield
Core 2 Duo Conroe
Itanium 2 McKinley
Cell
Core 2 Duo Wolfdale 3M
Pentium 4 Prescott-2M
Core 2 Duo Allendale
Pentium 4 Cedar Mill
AMD K8
Pentium 4 Prescott
Pentium 4 Northwood
Barton
Atom
Pentium 4 Willamette
Pentium III Tualatin
Pentium II Mobile Dixon
ARM Cortex-A9
AMD K7
Pentium III Coppermine
AMD K6-III
AMD K6
Pentium III Katmai
Pentium II Deschutes
Pentium Pro
Pentium II Klamath
Pentium
AMD K5
SA-110
Intel 80486
R4000
TI Explorer's 32-bit Lisp machine chip
ARM700
Intel 80386
Intel i960
ARM 3
Motorola 68020
DEC WRL MultiTitan
Intel 80286
ARM 9TDMI
Motorola 68000
Intel 80186
Intel 8086
Intel 8088
ARM 2
ARM 1
ARM 6
WDC 65C816
Novix NC4016
TMS 1000
Zilog Z80
Motorola 6809
WDC 65C02
RCA 1802
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Intel 8080
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Motorola 6800
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2020
Data source: Wikipedia (wikipedia.org/wiki/Transistor_count)
Year in which the microchip was first introduced
OurWorldinData.org – Research and data to make progress against the world's largest problems.
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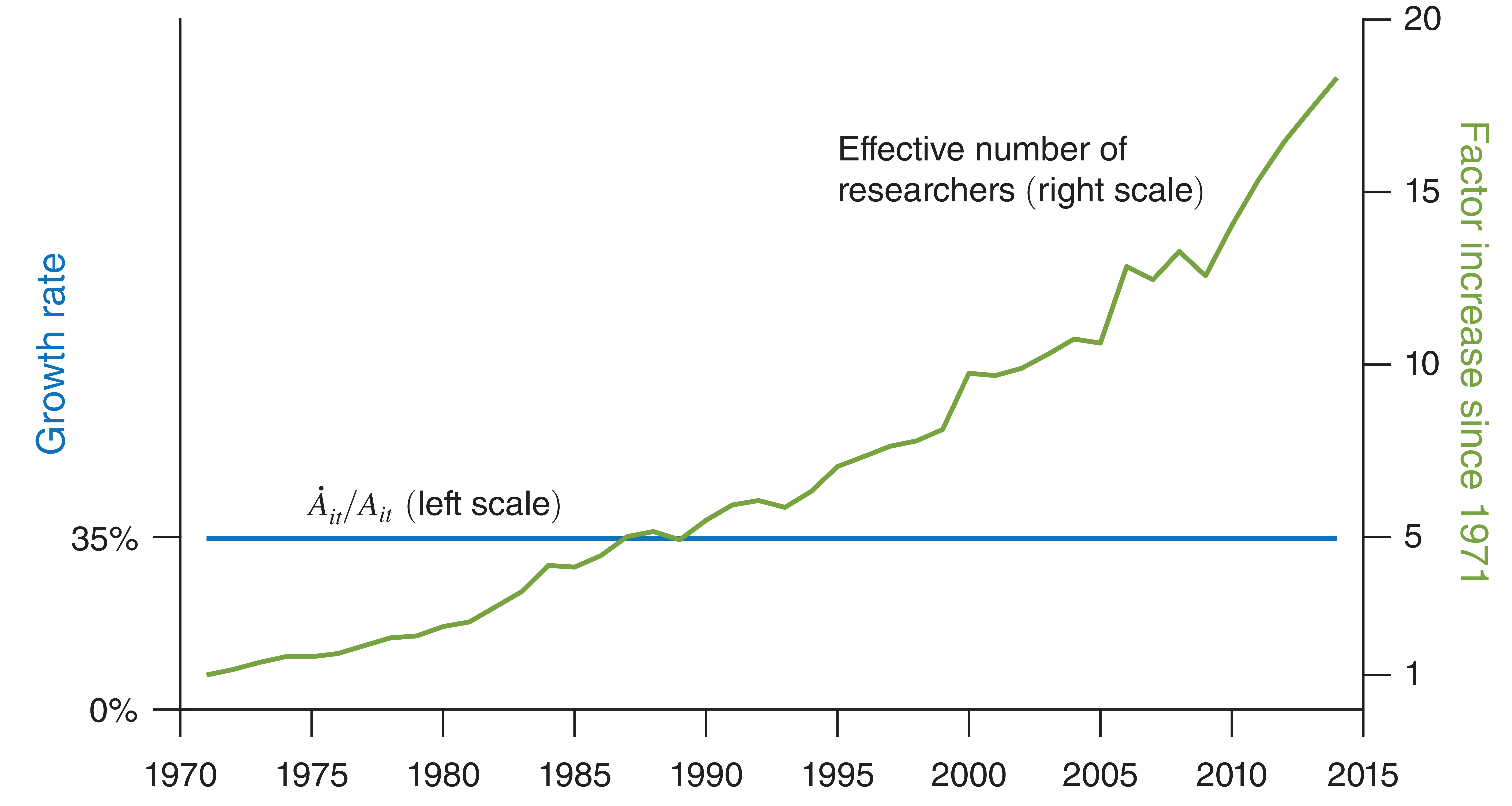
Researchers and TFP Growth: Moore's Law



Source: Bloom, Jones, Van Reenen, & Webb (2020)

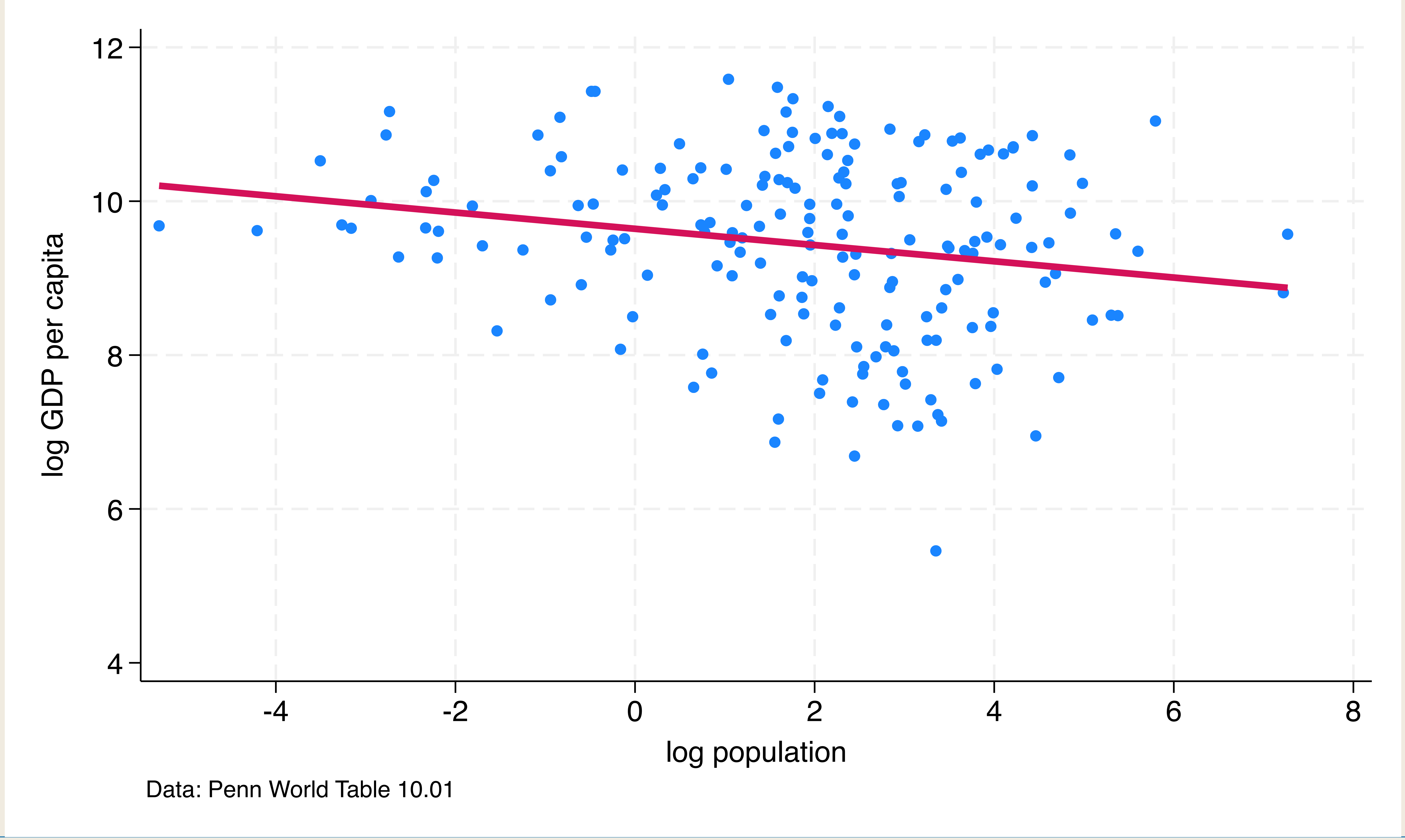
2. Does a Larger Population Size Raise Per-capita Income? – Peters (2022)

Population and Productivity

- Romer model strongly ties GDP to population
  - GDP grows faster if the population grows faster, $g_{Y/N} = \beta n/(1-\alpha)$
  - GDP level is higher if the population is larger, $Y_t/N_t = A_t^{\frac{\beta}{1-\alpha}} y$ and $A_t = s^R N_t/n$
- An increase in population raises productivity and income per capita
  … *holding everything else constant*
- Do we have any evidence?

Naive Idea

- What if we see the relationship between Y/N and N using cross-country data?

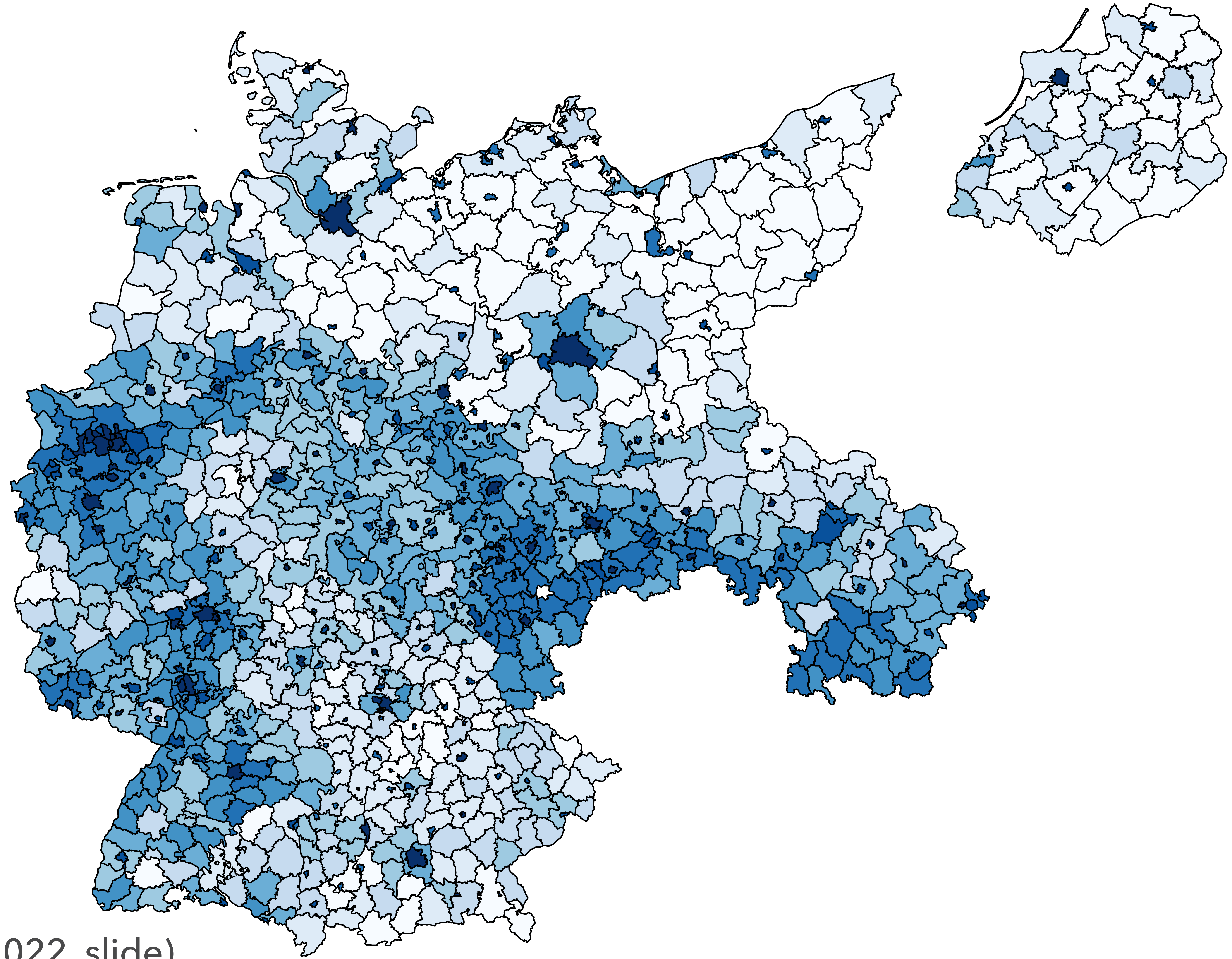


How Do We Isolate Population Size?

- The ideal thought experiment is that we **only** change the population size
- Countries differ not only in population size but a lot of other things…
- Peters (2022):

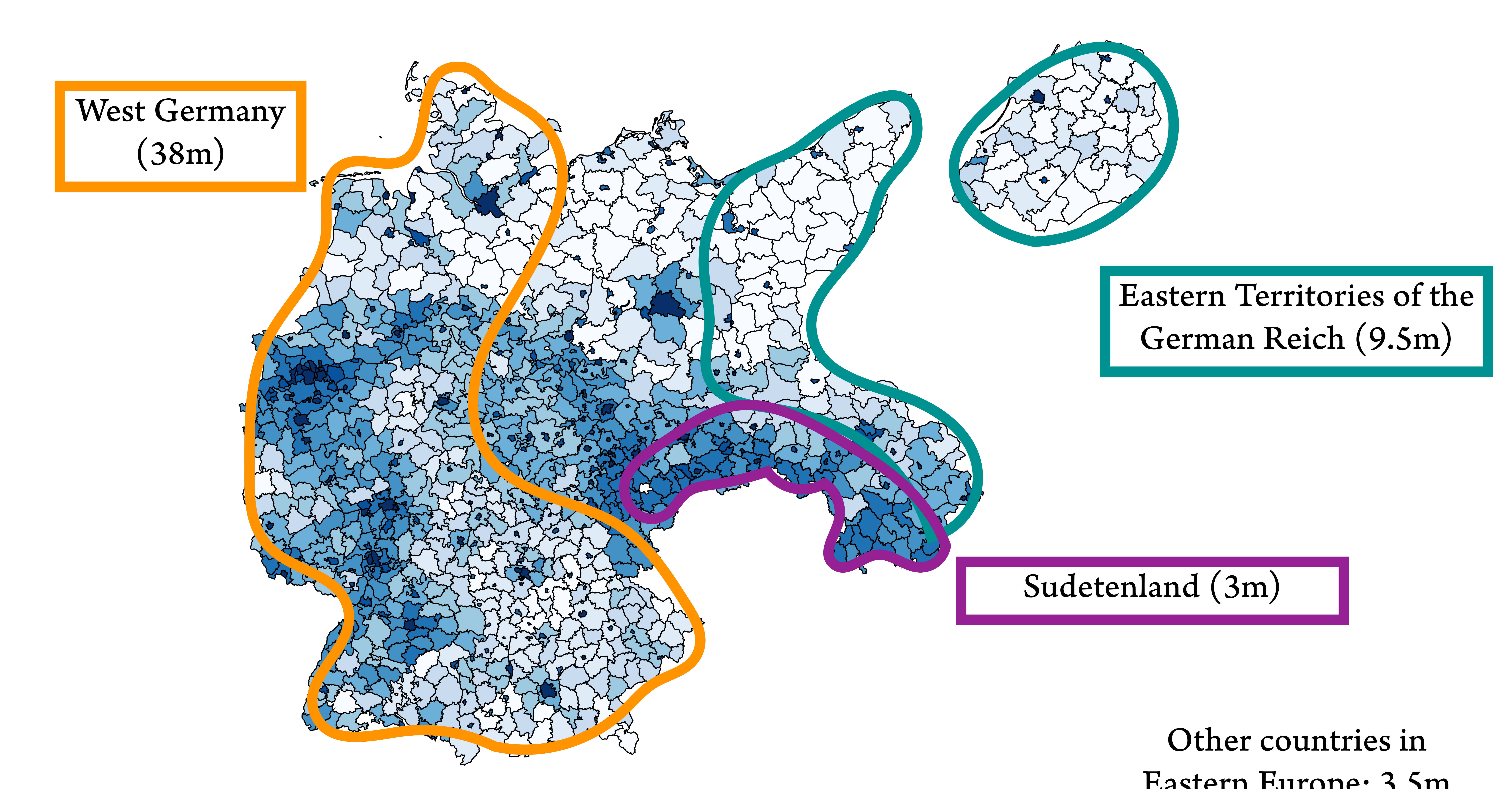
  Population expulsions in Germany after WW2 provide an ideal experiment

Germany in 1939



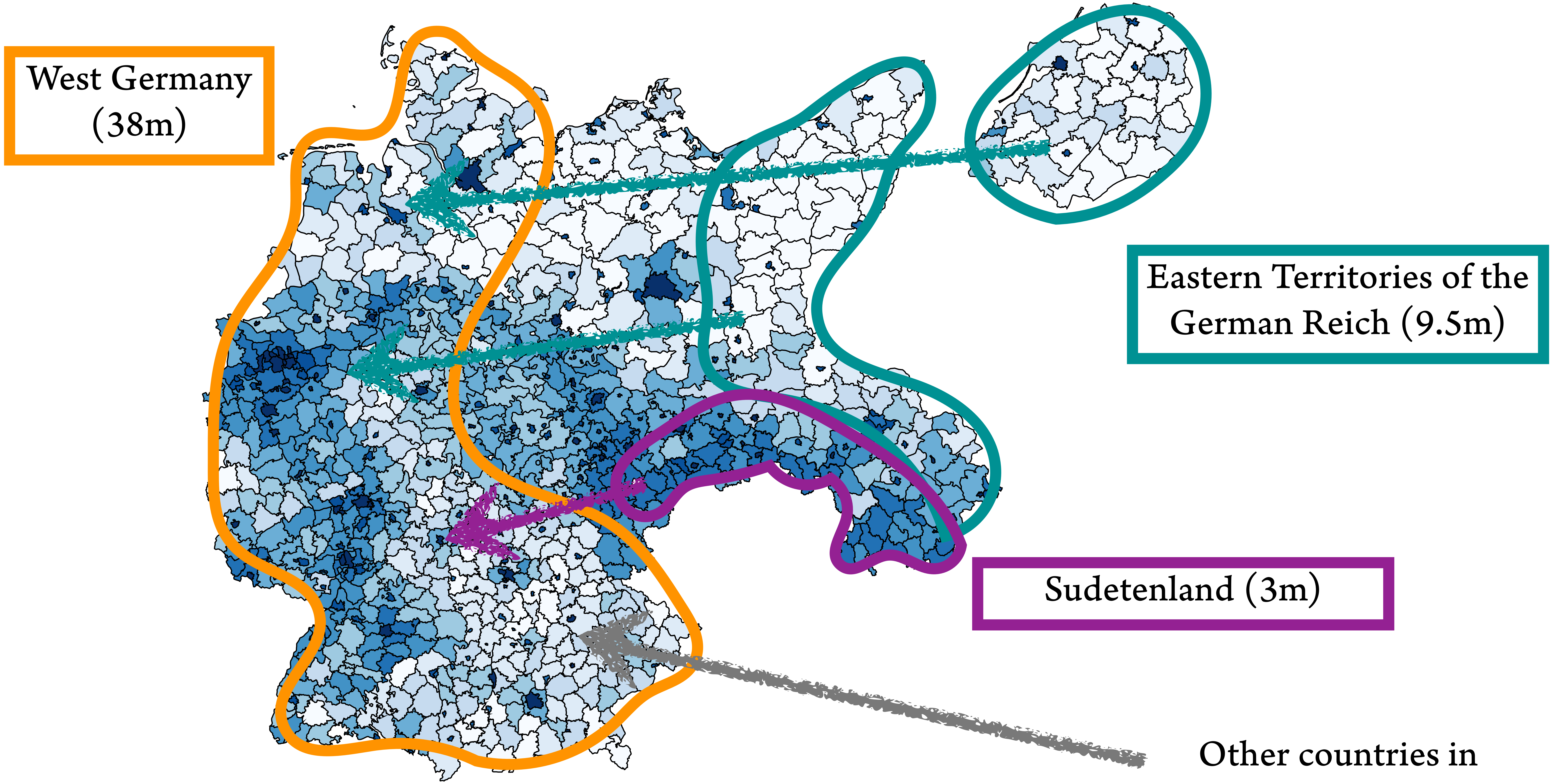
Source: Peters (2022, slide)

Distribution of German Ethnicity



Source: Peters (2022, slide)

The Expulsions: 1945 - 1949

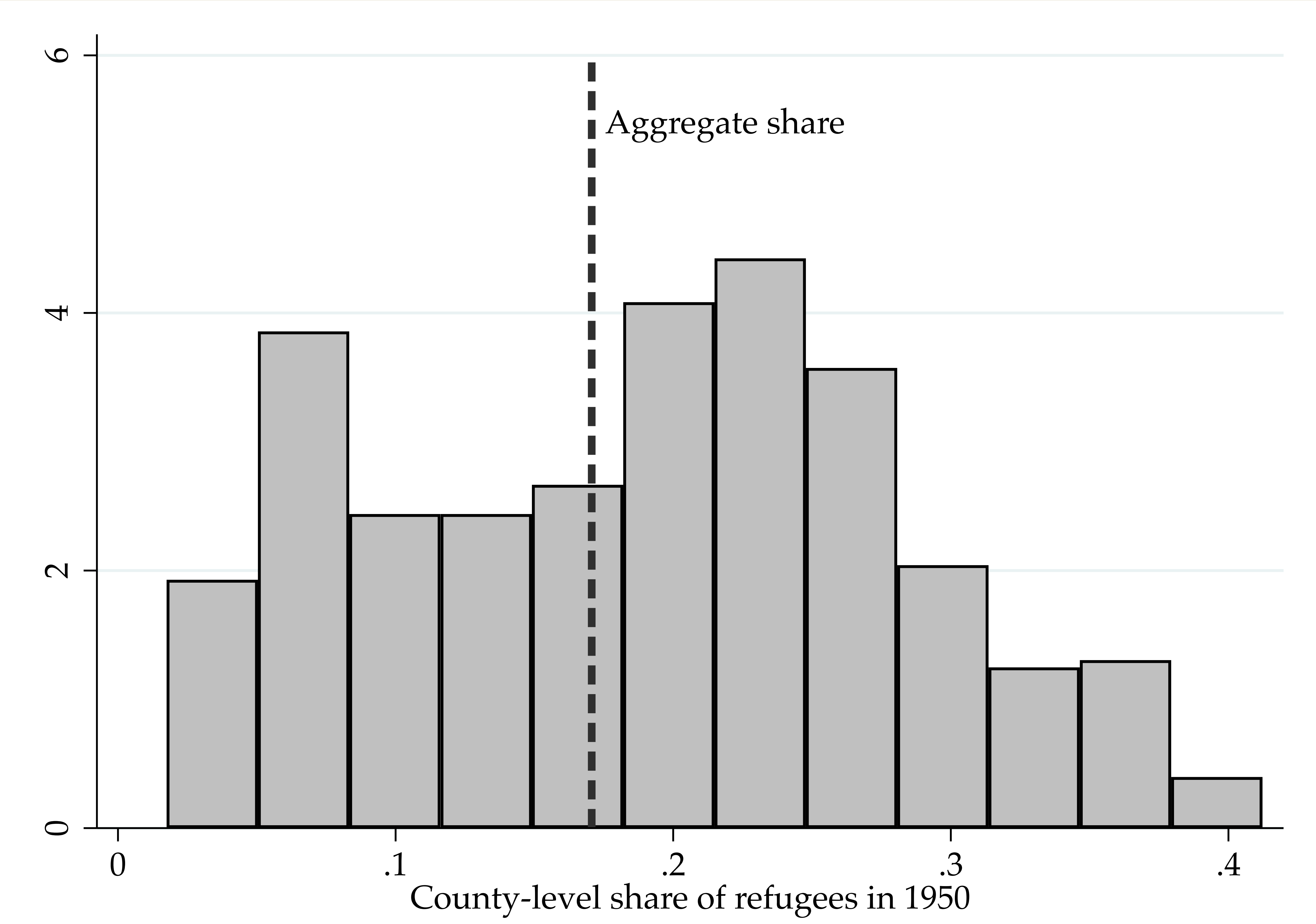


Source: Peters (2022, slide)

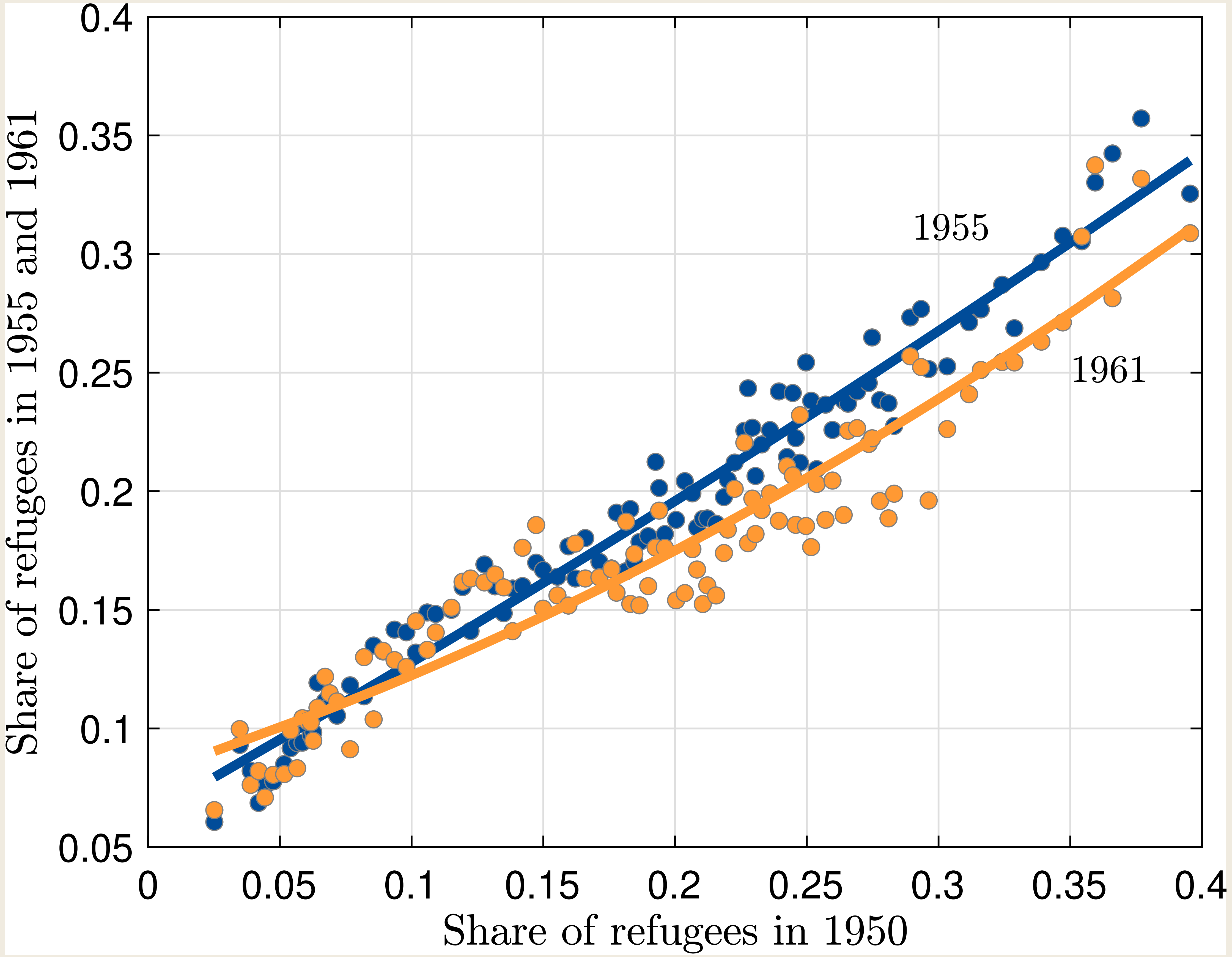
The Expulsions: 1945 - 1949

- Phase 1 (Nov 44 - Oct 45): **2m**
  - Expulsions / flight during the war
  - “Wild expulsions” after armistice
- Phase 2 (Jan 46 - July 1949): **6m**
  - Organized population transfers (Potsdam conference)
- West German population increased by 20% between 1939 and 1950

Heterogeneity and Persistence of Settlement



Source: Peters (2022)

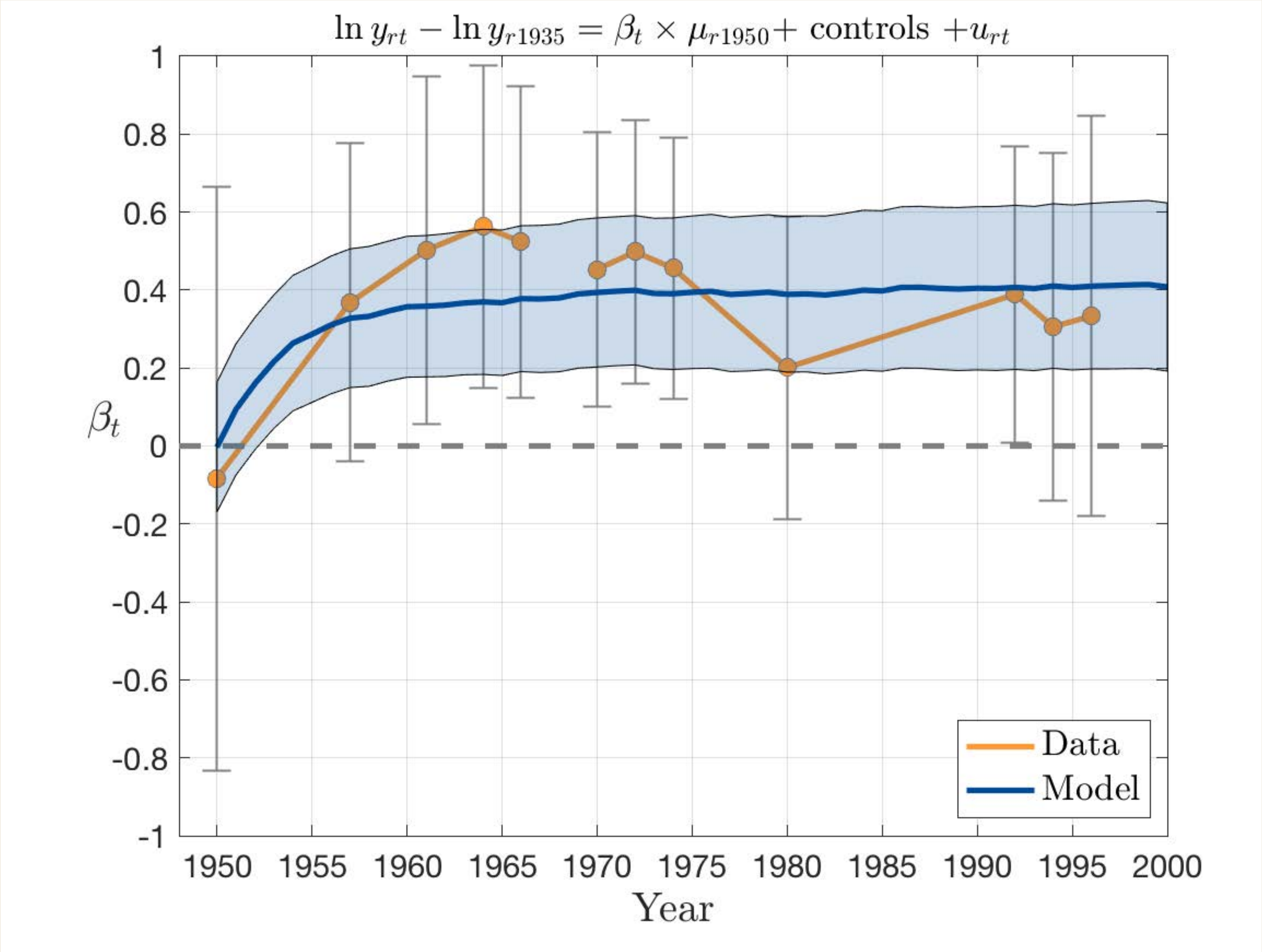


- The allocation of initial settlement of refugees
  1. varied dramatically across counties
  2. had a persistent effect

Question

- Allocation of refugees is mostly based on housing and food availability
- Ask:
  Did a county receiving a lot of refugees grow more
  … compared to a country receiving no (or few) refugees?

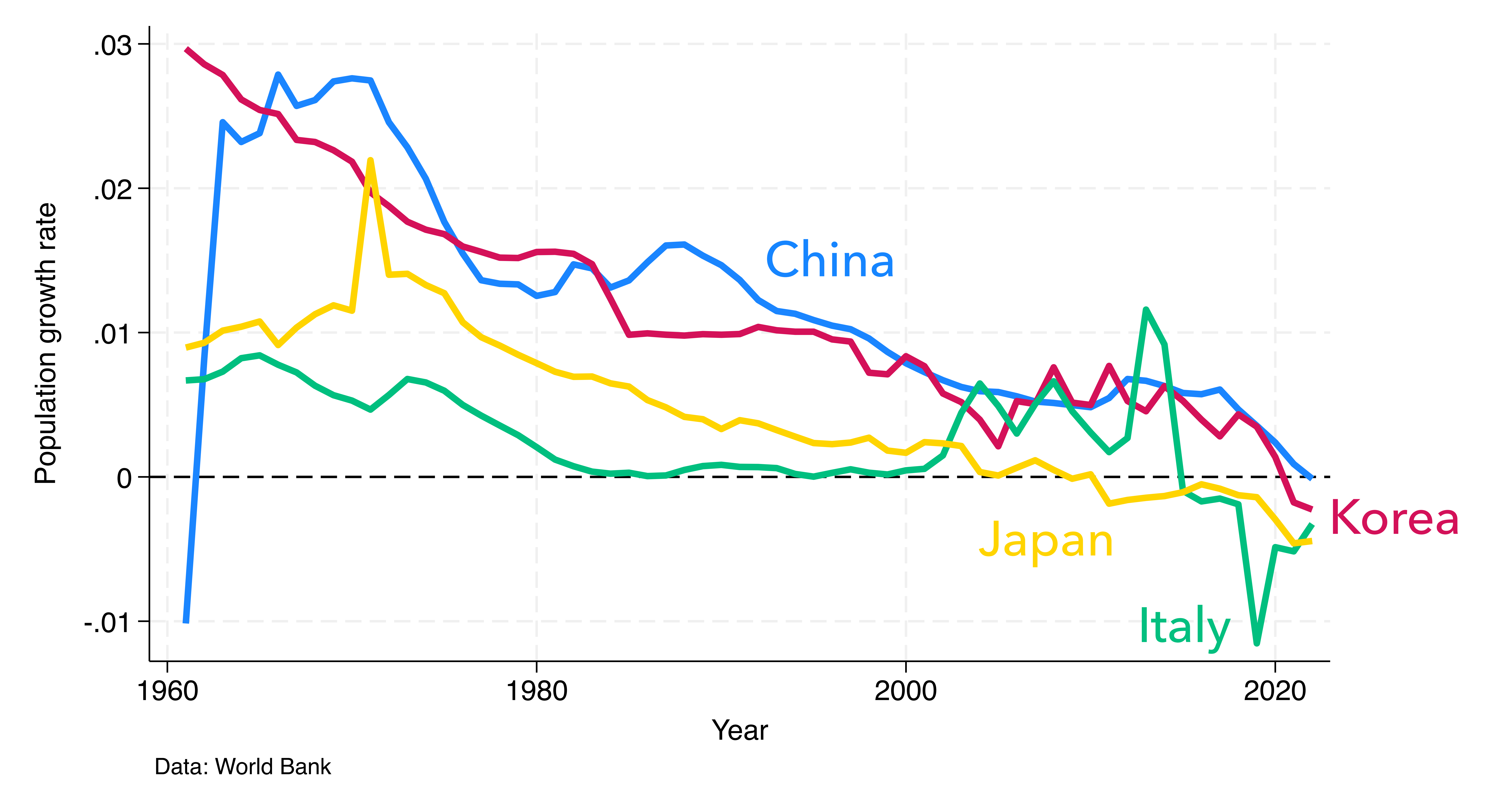
GDP Per Capita Increases with More Refugees



Source: Peters (2022, slide)

3. End of Economic Growth?
– Jones (2022)

Negative Population Growth



Future of Economic Growth

- Romer model predicts population growth is the engine of long-run growth
- Many countries already have negative population growth
  … and many others are predicted to be so in the next decades
- What do they mean for economic growth?

Negative Population Growth in Romer Model

- Let us go back to Romer model in the very beginning

$$A_{t+1} = A_t + s^R N_t \qquad (a)$$

- Population has a negative growth rate:

$$N_{t+1} = (1-\eta)N_t, \quad \eta > 0 \qquad (b)$$

- Iterating (b),

$$N_t = (1-\eta)^t N_0 \qquad (c)$$

- Plugging (b) into (c),

$$A_{t+1} = A_t + s^R (1-\eta)^t N_0 \qquad (d)$$

What Happens in the Long-run?

- Iterating (d),

$$A_{t+1} = A_0 + \sum_{s=0}^{t} s^R (1 - \eta)^s N_0$$

- In the long-run (as $t \to \infty$),

$$A_t \to A = A_0 + \frac{s^R}{\eta} N_0$$

- Since $Y_t = A_t^\beta L_t$ and $L_t = (1 - s^R)N_t$, GDP per capita is

$$Y_t / N_t = A_t^\beta (1 - s^R) \to A^\beta (1 - s^R) \quad \text{as} \quad t \to \infty$$

Empty Planet?

With negative population growth…

1. Knowledge stock converges to a constant
2. GDP per capita converges to a constant as well
   $\Rightarrow$ no economic growth
3. Population keeps declining, so total GDP keeps declining and converges to zero