Efficiency in Job-Ladder Models*

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Abstract

This paper examines the efficiency of a decentralized equilibrium in a broad class of random-search job-ladder models. We decompose the source of inefficiency into two margins: (i) investment margin, that is, the difference between the private and social benefit of job creation given the surplus of a match, and (ii) valuation margin, that is, the difference between the private valuation and the social valuation of a match surplus. In the presence of on-the-job searches, the well-known Hosios condition no longer guarantees these two margins align. Instead, the decentralized equilibrium with the Hosios condition features the excess creation of vacancies in the steady state and the excess volatility in unemployment in response to productivity shocks. Quantitatively, we find a significant difference between the equilibrium and the efficient allocation under standard calibration. Positive, regressive, and pro-cyclical taxes on jobs restore the efficiency of the equilibrium.

Keywords: search and matching, job ladder, on-the-job search, Hosios condition

JEL Classification: D61, D62, E24, J62, J63, J64

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1 Introduction

In designing and evaluating macroeconomic policies under frictional labor markets, understanding the normative properties of the equilibrium outcome is essential. In a frictional environment, evaluating the efficiency of decentralized equilibrium is nontrivial. For example, the mere presence of unemployment is not necessarily a sign of inefficiency, because matching workers and firms is costly.

Many recent studies highlight the importance of job-to-job transitions in understanding the positive aspects of the labor market at both the micro and the macro level. The commonly employed approach in modeling the job-to-job transition considers the *job ladder*. That is, workers move up to better and better jobs by conducting on-the-job searches and accepting better job offers. At the micro level, job ladders are important determinants of earnings growth and workers' career development.[1] At the macro level, job ladders shape the dynamics of aggregate productivity, wages, inflation, and unemployment.[2]

Despite their popularity, the normative properties of the job-ladder models are not well understood. In this paper, we examine the efficiency of a broad class of job-ladder models with random search. As in the standard Diamond-Mortensen-Pissarides (DMP) model, firms match with workers by posting vacancies. The probability of a match is governed by the matching function, which produces matches from the inputs of vacancies and searching workers. Jobs differ in match quality, and workers can engage in on-the-job searches. Consequently, employed workers gradually find a better job and move up the job ladder.

Our starting point is the well-known result by Hosios (1990). Hosios shows that in the standard DMP model without on-the-job search, an efficient outcome is achieved by the market equilibrium with Nash bargaining if the elasticity of the

[1] For example, Topel and Ward (1992) attribute about one-third of earnings growth for young workers to job-to-job transitions. More recently, Hahn et al. (2021) show workers (on average) gain both wages and hours with job-to-job transitions.

[2] For example, reallocating workers across jobs and improving allocation (Barlevy, 2002), generating wage dispersion (Postel-Vinay and Robin, 2002; Hornstein et al., 2011), inflation (Moscarini and Postel-Vinay, 2023; Faccini and Melosi, 2023; Birinci et al., 2024), wage rigidity (Fukui, 2020), unemployment dynamics (Moscarini and Postel-Vinay, 2018; Faberman et al., 2022), and aggregate productivity dynamics (Mukoyama, 2013).

matching function coincides with the worker's bargaining power (the Hosios condition). As Hosios points out, several inefficiencies exist in the DMP-style search-and-matching model. The Hosios condition ensures that these inefficiencies cancel out to achieve efficiency.

Building on the insight of Hosios, we organize these inefficiencies into two margins and call them the *investment margin* and *valuation margin*. The DMP model treats the job creation as a firm's investment. Firms invest in vacancies by paying the vacancy-posting cost and receive returns in the form of profit. On the investment margin, the question is whether the firm's incentive to invest, given the reward upon matching, is aligned with the social benefit of the investment activity. A separate question, on what we call the valuation margin, is whether the firm's reward upon matching is aligned with the social value of creating a match, that is, the value of moving a worker from unemployment to employment. In the standard DMP model without on-the-job search (Pissarides, 1985), the Hosios condition (coincidentally) ensures efficiency in both margins—killing two birds with one stone.

We show the Hosios condition does not guarantee efficiency once on-the-job search is taken into account. On the investment margin, the presence of on-the-job search implies forming a new match comes at the social cost of losing the current match. This cost is not necessarily internalized by firms when creating a vacancy, leading to a "worker-stealing" externality, similar to the business-stealing externality in the economic growth literature. We show imposing the sequential-auction wage-setting protocol of Cahuc et al. (2006) together with the Hosios condition ensures efficiency on the investment margin.

The same condition, however, does not ensure efficiency on the valuation margin. The social value of a job concerns how much searching on the job rather than off the job changes the congestion in the matching market. By contrast, the private value of a job concerns how much searching on the job rather than off the job creates the private surplus. For any value of the bargaining parameter, we have no reason to expect these two to coincide. For example, near the top of the job ladder, the private benefit of searching for another job is small (zero at the top), although the social cost of congesting the labor market is as large as another person with the same search intensity.[3]

We build on the above observations to analytically show that, under the Hosios condition, the decentralized equilibrium features the excessive creation of vacancies in the steady state. We further find the discrepancy is quantitatively significant under the standard calibration in the literature, which assumes both the Hosios condition and sequential auction. The decentralized equilibrium allocation has an unemployment rate that is about 1 percentage point lower than the efficient allocation. Furthermore, the excess creation of vacancies implies workers climb the job ladders too quickly, and the decentralized equilibrium has too many good matches compared with the efficient allocation. This finding is in contrast to a model with ex-ante job heterogeneity à la Acemoglu (2001) that has too few good jobs in the decentralized equilibrium.

We then show that, under the same calibration, the discrepancy between the decentralized equilibrium and the efficient allocation is further exacerbated in response to productivity shocks. The decentralized equilibrium features too much volatility in unemployment compared with the efficient allocation. The reason is that the decentralized equilibrium overvalues jobs more significantly during the labor market booms.

These results are important for applied work because when a job-ladder model is calibrated to satisfy the Hosios condition, which is a common practice in the literature, the model automatically favors a policy that suppresses vacancy creation in the steady state, such as income taxes and unemployment insurance, and stabilizes labor market fluctuations, such as monetary and fiscal policies, even without any other frictions.

In the final part of the paper, we show a combination of output taxes and entry taxes restore the efficiency of the equilibrium. The output tax corrects inefficiency along the valuation margin, and the entry tax corrects the inefficiency along the investment margin. Under standard calibration assuming the Hosios condition and sequential auction, we find the output tax needs to be positive, regressive at the top, and pro-cyclical. By contrast, the entry tax is zero.

[3] The search intensity is exogenous in our baseline model, but even when the search intensity is endogenous, the social cost is not taken into account when workers make their search decision.

Related literature

Our paper is related to several strands of the literature. The most directly related strand includes the papers that examine efficiency in the models with on-the-job search. An earlier paper by Gautier et al. (2010) is closely related. Gautier et al. (2010) also analyze the efficiency of matching models with on-the-job searches and costly vacancy creation. As we show, some of their intuitions carry over to our model. However, they use a model with a structure (a "circular" heterogeneity) that is not typically used in macroeconomics. They focus on a particular case where on-the-job search and off-the-job search have the same efficiency and the discount rate approaches zero. Our model has a structure that is more commonly seen in the macroeconomic literature, and we consider wage-setting protocols that are popularly used in applied quantitative literature. Cai (2020) mainly considers a discrete-time model where multiple firms can match with a worker within a period. His focus is on deriving efficiency conditions (which generally involve endogenous variables), and he also shows the traditional Hosios condition does not necessarily guarantee efficiency in the presence of an on-the-job search. We focus more on characterizing the source and nature of inefficiency, particularly in the model close to the standard DMP model.

The second strand of literature is the analysis of the DMP model with heterogeneous jobs. Papers such as Acemoglu (2001), Davis (2001), and Mukoyama (2019) explicitly consider efficiency in DMP models with heterogeneous jobs and no on-the-job search. These papers do not deal with job-to-job transitions.

Finally, previous studies have examined the efficiency of equilibrium in directed-search settings. For example, Menzio and Shi (2011) establish the efficiency of directed-search equilibrium in a model with on-the-job search. Our paper considers random-search models, where different types of workers interact in the same labor market.

This paper is organized as follows. Section 2 sets up the terminology we use in our analysis by reviewing the intuition of the Hosios (1990) condition in a model without on-the-job searches. Section 3 sets up our model and compares the efficient allocation and the equilibrium outcome. In Section 4, we derive some analytical results. Section 5 examines the model quantitatively. Section 6 analyzes how efficient allocation can be implemented as a market equilibrium with taxes. Section 7 concludes.

2 Revisiting the basic intuition of the Hosios condition

Before considering our baseline model, to develop basic intuition, let us set up the model without an on-the-job search. Consider the textbook Pissarides (1985) model in continuous time. A continuum of infinitely-lived workers with population 1 exists in the economy. The workers are risk neutral, and the discount rate is $r > 0$. The workers are either employed or unemployed. For a worker to be employed, she has to be matched with a firm. Production takes place with a one-worker, one-firm match. The match produces $z > 0$ units of the consumption good. An unemployed worker receives the flow value of $h \in [0, z)$ from home production. Firms post vacancies to fill their positions. The flow vacancy cost is $\kappa > 0$. We assume free entry: any firm can post a vacancy and start producing once matched with a worker.

In this section, on-the-job search does not exist; only unemployed workers look for jobs. We assume the matching is random, and therefore, all vacancies have the same chance to match with a worker, and all unemployed workers have the same chance to match with a vacancy. The matching process is governed by the matching function: $M(u, v)$ represents the number of matches created when v vacancies and u unemployed workers exist. The matching function satisfies the following conditions: (i) $M(u, v)$ is strictly increasing and strictly concave in each of u and v and satisfies Inada conditions; (ii) $M(u, v)$ exhibits constant-returns to scale; and (iii) $M(u, v) \leq \min\{u, v\}$. We assume the separation is random with the Poisson probability $\sigma > 0$. In the market equilibrium, the wages are set following the generalized Nash bargaining solution. Nash bargaining is conducted on the expected present value of surpluses, and the worker's bargaining power is set at $\gamma \in [0, 1)$. Because of the random-matching assumption, the Poisson rate at which an unemployed worker finds a job and a vacancy finds a worker can be written as

$$p(\theta) \equiv M(1, \theta)$$

and

$$q(\theta) \equiv M\left(\frac{1}{\theta}, 1\right),$$

where $\theta \equiv v/u$ is the market tightness. Note $p(\theta) = \theta q(\theta)$ holds.

The details of the social planner's problem and the market equilibrium are analyzed in Appendix A. It shows the Hosios condition (Hosios, 1990)

$$\gamma = \eta(\theta), \tag{1}$$

where $\eta(\theta) \equiv -\theta q'(\theta)/q(\theta)$ is the elasticity of the matching function, ensures the constrained efficiency of the market outcome. That is, under condition (1), the solution to the social planner's problem coincides with the market equilibrium. Intuitively, the Hosios condition can be understood as the condition where two inefficiencies exactly cancel out each other.

Let us go over the intuition more closely. For the ease of exposition, we focus on the steady state. First, consider the optimization problem for vacancy posting. This exercise amounts to comparing between

$$\kappa = (1 - \eta(\theta))q(\theta)\mu \tag{2}$$

for the social planner's problem, where μ is the social value of moving one unemployed worker into employment, and

$$\kappa = (1 - \gamma)q(\theta)S, \tag{3}$$

where S is the surplus of a match between a worker and a firm. Here, suppose $\mu = S$ holds. Even in that situation, two inefficiencies arise in comparing equations (2) and (3). We call this margin the "investment margin," because it relates to the firm's vacancy creation as an investment. The first inefficiency is the hold-up problem. Matches are formed because of the firms' active investment (vacancy posting), and workers do not incur any costs. Thus, all returns from the match should be paid to the firms to ensure an efficient level of investment. However, at the time the worker and the firm engage in Nash bargaining, the investment costs are already sunk. As a result, the firm can collect only $(1-\gamma)$ share of the surplus (the hold-up problem) on the right-hand side of (3). This inefficiency, due to firms' imperfect appropriation of surplus by firms, leads to too few vacancies compared with the socially desirable level. When γ (the worker's bargaining power) is large, the inefficiency is large.

The second inefficiency is the matching externality. In general, a firm posting a vacancy generates externalities to both workers and firms. On the worker side, an increased vacancy raises the probability of an unemployed worker finding a job. This externality does not lead to inefficient allocation here, because the workers do not make a decision. On the firm side, the increase in vacancy by one firm makes the matching of the other firms difficult due to congestion. The firm does not take into account this congestion externality, and thus, the outcome is too many vacancies. To see why this externality is related to $\eta(\theta)$ in equation (2), consider the effect of a marginal increase in vacancies by one firm. From this firm's (private) perspective, the expected number of matches increases by $q(\theta) = M(u,v)/v$. From the social perspective, however, the increase in the match is $M_2(u,v)$, which is lower than $M(u,v)/v$ (recall that M is concave in each term). The difference $M_2(u,v) - M(u,v)/v$ represents the externality. It is straightforward to show that $M_2(u,v) - M(u,v)/v = -q(\theta)\eta(\theta)$, and thus, the term $-q(\theta)\eta(\theta)$ represents the (negative) externality each vacancy creation generates.

Now, consider how the inefficiency shows up in the calculation of the social and private values of the match. We call this margin the "valuation margin." This exercise compares

$$(r+\sigma)\mu = z - h - (p(\theta)\mu - \kappa\theta) \tag{4}$$

and

$$(r+\sigma)S = z - h - p(\theta)\gamma S. \tag{5}$$

In equation (4), the (flow) social value of the match is the sum of the current surplus $z-h$ and the opportunity cost of keeping a worker employed. The opportunity cost is that, by keeping the worker unemployed, she could have generated a new match with probability $p(\theta)$. However, this calculation does not take into account that increasing u changes θ if v is kept constant. To keep θ constant, v has to be increased by θ units (because $(v+\theta)/(u+1) = v/u$). These additional vacancies would cost $\kappa\theta$, which is subtracted from the gain from matching. Another way of thinking about the second term is the match generated by the marginal (unemployed) worker $M_1(u,v)$ times the value of the match μ. This fact can be seen from $p(\theta)\mu - \kappa\theta = p(\theta)\mu - \theta q(\theta)(1-\eta(\theta))\mu = \theta q(\theta)\eta(\theta)\mu$, which is equal to $M_1(u,v)\mu$. In the market equilibrium equation (5), the second term on the right-hand side, $p(\theta)\gamma S$, represents the worker's opportunity cost from working (not searching). The full opportunity cost is $p(\theta)S$, but here the fraction $p(\theta)(1-\gamma)S$ is unaccounted for because the worker only receives a γ fraction of the surplus.

A similar logic as above holds: the worker's private value looks at $p(\theta) = M(u,v)/u$, whereas the social value is $M_1(u,v)$—the private value does not take into account the negative externality imposed on the other workers. The private value corresponds to $p(\theta)\mu$, and the externality term is $\kappa\theta$ (it is the amount of resources required to "undo" the externality). Under the Hosios condition, this inefficiency is offset by the inefficiency that the worker only recognizes the γ fraction of the opportunity cost (the surplus it could have created by being unemployed).

As we can see, the Hosios condition achieves "offsetting one inefficiency by another inefficiency" on the firm side as well as the worker side. It can "kill two birds with one stone," because both inefficiencies are symmetric: the firms and the unemployed workers create similar externalities, and they are on the opposite sides of the bargaining. Moreover, the direction of the total inefficiencies is aligned when the Hosios conditions do not hold: when γ is too small, S is too large, which implies $(1-\gamma)S$ is too large, because (i) $(1-\gamma)$ is too large and (ii) S is too large. Thus, θ is too large in equilibrium.

3 The model with on-the-job search

In this section, we present our main model. The model now features on-the-job search.

3.1 Market equilibrium

The economy is populated by a unit mass of workers. All workers have the following linear preferences:

$$\int_0^\infty e^{-rt} c_t dt,$$

where $r > 0$ is, once again, the discount rate, and c_t is the consumption at t. Workers receive wages (and consume them) while working and enjoy h units of consumption while unemployed.

Here, the new assumption is that the workers search both off and on the job. That is, matches are heterogeneous, and an employed worker may meet with a new job (new match). When the new job is better, the worker moves to the new job. Thus, over time, a worker may climb up the job ladder.

Firms create vacancies at per-vacancy cost $\kappa > 0$. When firms meet with workers, they draw permanent match quality, $z \in [0, \infty)$, with pdf $g(z)$ and cdf $G(z)$. We assume the match quality has a finite mean. Firms with match quality $A_t z$ produce z units of output in each period, where A_t is the aggregate productivity.

The economy is subject to matching friction. We normalize the search efficiency of unemployed workers to one. Employed workers have a search intensity of $\zeta \in [0, 1]$. The total efficiency of the search on the worker side is

$$x_t \equiv u_t + \zeta(1 - u_t).$$

We assume the matching is random. Let

$$f_t^u \equiv \frac{u_t}{x_t}, \quad f_t(z) \equiv \frac{\zeta n_t(z)}{x_t} \tag{6}$$

be the probability of a vacancy encountering unemployed workers and the probability density a vacancy encountering employed workers with match quality z conditional on the meeting, respectively. Here, $n_t(z)$ denotes the measure of workers employed at z.

Given the number of vacancies created (v_t), the number of matches in the economy is given by the matching function $M(x_t, v_t)$, where we assume this function is increasing in both terms and exhibits constant returns to scale. We denote

$$\theta \equiv \frac{v_t}{x_t}$$

as the labor market tightness. The Poisson rate that an unemployed worker meeting a vacancy and a vacancy meeting a worker is $p(\theta_t) \equiv M(1, \theta_t)$ and $q(\theta_t) \equiv M(1/\theta_t, 1)$, respectively. The meeting rate of workers employed in z is $\zeta p(\theta_t)$. The match separates at rate $\sigma > 0$.

We let U_t denote a worker's value of being unemployed, $E_t(z, \bar{O})$ denote the value of an employed worker with match quality z and the worker's outside option $\bar{O}$, $J_t(z, \bar{O})$ denote the value of a firm with match quality z and the worker's outside option $\bar{O}$, and V denote the value of a vacancy. Define the joint match surplus to be

$$S_t(z) \equiv W_t(z, \bar{O}) + J_t(z, \bar{O}) - U_t - V_t, \tag{7}$$

where $S_t(z)$ is independent of the worker's outside option and increasing in z, which we confirm below.

The wages are determined by Nash bargaining, but the worker's outside option potentially depends on the past history. When unemployed workers decide to form a match, they engage in Nash bargaining with the worker's bargaining weight $\gamma \in [0, 1]$. The outside option for unemployed workers is the value of unemployment, U_t.

When employed workers at firm z with outside option $\bar{O}$ meet with poachers z', the following takes place. When $S_t(z) \geq S_t(z')$, workers stay and bargain with outside option $\max\{\bar{O}, U_t + \omega S_t(z')\}$, where $\omega \in [0, 1]$ is a parameter governing the degree of offer-matching and $S_t(z')$ is the joint match surplus of job z', which we define below. When $S_t(z) < S_t(z')$, workers are poached and the outside option of workers is given by $U_t + \omega S_t(z)$. The offer-matching parameter ω is flexible enough to nest many existing wage-setting protocols. Two important special cases are the following:

1. Nash bargaining with no commitment (e.g., McCrary, 2022): $\omega = 0$;

2. Sequential auction (e.g., Cahuc et al., 2006): $\omega = 1$.[4]

We will highlight these two cases later on.

[4] A subtle point is that here we assume the wage renegotiation occurs continuously, as in the standard DMP model, whereas in Cahuc et al. (2006), a negotiation occurs only when a new meeting happens. This difference does not result in different outcomes in the steady state.

The value function of the unemployed is

$$rU_t = h + p(\theta_t) \int g(z) \max\{W_t(z, U_t) - U_t, 0\} dz + \dot{U}_t, \tag{8}$$

where the "dot notation" represents the time derivative: $\dot{U}_t \equiv \partial U_t / \partial t$. Value function of workers employed in firm z with an outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned} rW_t(z, \bar{O}) = w_t(z, \bar{O}) & \\ & + \zeta p(\theta_t) \int_z^\infty g(z') \left(W_t(z', U_t + \omega S_t(z)) - W_t(z, \bar{O}) \right) dz' \\ & + \zeta p(\theta_t) \int_0^z g(z') \left(W(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - W(z, \bar{O}) \right) dz' \\ & + \sigma(U_t - W_t(z, \bar{O})) + \dot{W}_t(z, \bar{O}). \end{aligned} \tag{9}$$

The value of the filled job with productivity z and outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned} rJ_t(z, \bar{O}) = A_t z - w_t(z, \bar{O}) & \\ & + \zeta p(\theta_t) \int_z^\infty g(z') \left(V_t - J_t(z', \bar{O}) \right) dz' \\ & + \zeta p(\theta_t) \int_0^z g(z') \left(J_t(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - J_t(z, \bar{O}) \right) dz' \\ & + \sigma \left(V_t - J_t(z, \bar{O}) \right) + \dot{J}_t(z, \bar{O}). \end{aligned} \tag{10}$$

Given the above bargaining protocol, the outside option $\bar{O}$ cannot exceed $U + \omega S(z)$ for a match with productivity z along the equilibrium path. Thus we ignore such a possibility. The value of vacancy is given by

$$\begin{aligned} V_t = -\kappa & \\ & + f_t^u q(\theta_t) \int_0^\infty g(z) \max\{J_t(z, U) - V_t, 0\} dz \\ & + q(\theta_t) \int_0^\infty g(z) \int_z^\infty f(z') \max\{J_t(z, U + \omega S_t(z')) - V_t, 0\} dz' dz + \dot{V}_t, \end{aligned} \tag{11}$$

where $\kappa > 0$ is the vacancy-posting cost. We assume free entry into vacancy posting. This assumption implies zero value from vacancy posting:

$$V_t = 0. \tag{12}$$

The Nash bargaining with worker's outside option $\bar{O}$ solves

$$\max_{w_t(z,\bar{O})} \left(W_t(z,\bar{O}) - \bar{O}\right)^{\gamma} \left(J_t(z,\bar{O}) - V_t\right)^{1-\gamma}.$$

After imposing the free-entry condition (12), the solution to the bargaining problem yields

$$W_t(z,\bar{O}) = \bar{O} + \gamma(S_t(z) - (\bar{O} - U_t)) \tag{13}$$

and

$$J_t(z,\bar{O}) = (1-\gamma)(S_t(z) - (\bar{O} - U_t)). \tag{14}$$

Plugging (8), (9), and (10) into (7) and using (13), we obtain the recursive expression for the job-match surplus:

$$\begin{aligned}(r+\sigma)S_t(z) = A_t z - h& \\ &+ \zeta p(\theta) \int_z^{\infty} g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right) dz' \\ &- p(\theta_t) \int_{\underline{z}_t}^{\infty} g(z')\gamma S_t(z') dz' + \dot{S}_t(z).\end{aligned} \tag{15}$$

The reservation match quality $\underline{z}_t$, above which the match is formed, satisfies

$$S_t(\underline{z}_t) = 0. \tag{16}$$

The expression (15) confirms our original presumption that the joint match surplus is independent of the worker's outside option $\bar{O}$.

After imposing free-entry condition (12) and also (14) in (11), we obtain

$$\kappa = (1-\gamma)q(\theta_t) \left[f_t^u \int_{\underline{z}} g(z) S_t(z) dz + \int_0^{\infty} \int_z^{\infty} f_t(z) g(z') \left[S_t(z') - \omega S_t(z)\right] dz' dz \right]. \tag{17}$$

We denote the mass of workers employed at match quality below z as $N_t(z)$. Its law of motion satisfies

$$\dot{N}_t(z) = (G(z) - G(\underline{z}_t))\, p(\theta_t) u_t - N_t(z)(1 - G(z))\zeta p(\theta_t) - \sigma N_t(z), \tag{18}$$

with a boundary condition $N_t(\underline{z}_t) = 0$. By its definition,

$$\partial_z N_t(z) = n_t(z), \tag{19}$$

where $\partial_z N_t(z) \equiv \partial N_t(z)/\partial z$. The law of motion of unemployment over the time interval dt is

$$du_t = [-p(\theta_t)(1 - G(\underline{z}_t)) + \sigma(1 - u_t)]\, dt + dN_t(\underline{z}_t), \tag{20}$$

where $dN_t(\underline{z}_t)$ is the mass of workers endogenously separating over the time interval dt.

We are now ready to define equilibrium.

Definition 1 *Given $\{N_0(z), u_0\}$, the equilibrium consists of a sequence of the joint match surplus $\{S_t(z)\}$, the employment distribution across the job-ladder, $\{N_t(z), n_t(z), f_t(z), f_t^u\}$, and the unemployment rate, $\{u_t\}$, market tightness, $\{\theta_t\}$, and reservation match quality $\{\underline{z}_t\}$ such that (6), (7), (16), (17), (18), (19), and (20) hold. The steady-state equilibrium is the one where all variables are constant over time.*

3.2 Social planner's problem

The social planner directly controls the offer-acceptance decisions of all workers and thereby all worker flows subject to matching frictions as well as vacancy creation. Let $\mathbb{I}_t^{UE}(z)$ be an indicator function that takes a value of 1 if an unemployed worker meeting a job with match quality z accepts an offer. Likewise, let $\mathbb{I}_t^{EE}(z, z')$ be an indicator function that takes a value of 1 if an employed worker with match quality z meeting a job with match quality z' accepts an offer. Finally, ς_t denotes the mass of workers for whom the planner resolves the match at time t.

The social planner's problem is to choose $\{\theta_t, n_t(z), \mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z, z'), \varsigma_t\}$ to maximize

$$\int_0^\infty e^{-rt} \left[\int_0^\infty A_t z n_t(z) dz + h \left(1 - \int_0^\infty n_t(z) dz \right) - \kappa \theta_t \left(1 - \int_0^\infty n_t(z) dz + \int_0^\infty \zeta n_t(z) dz \right) \right] dt$$

subject to

$$\begin{aligned} \dot{n}_t(z) = & \left(1 - \int n_t(z') dz' \right) p(\theta_t) g(z) \mathbb{I}_t^{UE}(z) + \int_0^\infty p(\theta_t) g(z) \mathbb{I}_t^{EE}(z', z) \zeta n_t(z') dz' \\ & - \int_0^\infty p(\theta_t) g(z') \mathbb{I}_t^{EE}(z, z') \zeta n_t(z) dz' - \sigma n_t(z) - \varsigma_t. \end{aligned} \tag{21}$$

The first term in the square brackets in the objective function is the production by active matches, the second term is the home production, and the third term is the vacancy-posting cost κv_t. For the constraint, the change in the number of matches with the match quality z is due to (i) matches created with workers moving from U to E plus (ii) E to E movements from another z' to z minus (iii) E to E movements out of z (to another z') and separation.

The current-value Hamiltonian for this problem is

$$H = \int_0^\infty A_t z n_t(z) dz + h\left(1 - \int_0^\infty n_t(z)dz\right) - \kappa\theta_t \left(1 - \int_0^\infty (1-\zeta) n_t(z) dz\right)$$
$$+ \int_0^\infty \mu_t(z) \left[\left(1 - \int_0^\infty n_t(z')dz'\right) p(\theta_t) g(z) \mathbb{I}_t^{UE}(z) + \zeta \int_0^\infty p(\theta_t) g(z) \mathbb{I}_t^{EE}(z',z) n_t(z') dz'\right.$$
$$\left. - \zeta \int_0^\infty p(\theta_t) g(z') \mathbb{I}_t^{EE}(z,z') dz' - \sigma n_t(z) - \varsigma_t \right] dz,$$

where $\mu_t(z)$ is the costate variable that represents the shadow value of the constraint (21). Thus, $\mu_t(z)$ is the shadow value of creating one unit of match with match quality z.

The optimality conditions for $\{\mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z,z')\}$ are

$$\mathbb{I}_t^{UE}(z) = \begin{cases} 1 & \mu_t(z) > 0 \\ 0 & \mu_t(z) \le 0 \end{cases}, \qquad \mathbb{I}_t^{EE}(z,z') = \begin{cases} 1 & \mu_t(z') > \mu_t(z) \\ 0 & \mu_t(z') \le \mu_t(z) \end{cases}.$$

The optimality condition for endogenous separation ς_t implies $\mu_t(z) \ge 0$ for all z with $n_t(z) > 0$.

The first-order optimality condition on $n_t(z)$ is

$$(r+\sigma)\mu_t(z) = A_t z - h - \int_{\underline{z}_t}^\infty p(\theta_t) g(z') \mu_t(z') dz' + \zeta p(\theta_t) \int_z^\infty g(z')(\mu_t(z') - \mu_t(z)) dz'$$
$$+ \kappa\theta_t(1-\zeta) + \dot{\mu}_t(z), \tag{22}$$

where we have already imposed the fact that $\mu_t(z)$ is increasing in z. The reservation match quality $\underline{z}_t$ satisfies

$$\mu_t(\underline{z}_t) = 0.$$

The first-order optimality condition for θ_t is

$$\begin{aligned}\kappa\left(1-(1-\zeta)\int_0^\infty n_t(z)dz\right) =&(1-\eta(\theta_t))q(\theta_t)\int_{\underline{z}}^\infty \mu_t(z)\left(1-\int_{\underline{z}_t}^\infty n_t(z')dz'\right)g(z)dz\\ &+(1-\eta(\theta_t))q(\theta_t)\zeta\int_0^\infty\int_0^\infty \mu_t(z)g(z)\mathbb{I}_t^{EE}(z',z)n_t(z')dz'dz\\ &-(1-\eta(\theta_t))q(\theta_t)\zeta\int_0^\infty\int_0^\infty \mu_t(z)g(z')\mathbb{I}_t^{EE}(z,z')n_t(z)dz'dz.\end{aligned} \tag{23}$$

where $\eta(\theta)\equiv -\theta q'(\theta)/q(\theta)$.

We can rewrite (23) as

$$\kappa=(1-\eta(\theta_t))q(\theta_t)\left[f_t^u\int_{\underline{z}_t}^\infty g(z')\mu(z')dz'+\int_{\underline{z}_t}^\infty\int_z^\infty f_t(z)g(z')\left(\mu_t(z')-\mu_t(z)\right)dz'dz\right], \tag{24}$$

where f_t^u and $f_t(z)$ are defined as (6). The left-hand side is the cost of posting one vacancy, and the right-hand side is the marginal increase in the matching probability $p'(\theta_t)=(1-\eta(\theta_t))q(\theta_t)$ times the value of the match the vacancy creates. With frequency f^u, the vacancy meets an unemployed worker and generates value $\mu_t(z')$ with probability (density) $g(z')$ (when $\mu_t(z')>0$). With frequency $f_t(z)$, it meets with an employed worker with quality z and draws the new match quality z' with density $g(z')$. When $\mu_t(z')>\mu_t(z)$, the worker moves to the new job, and the social value $\mu_t(z')-\mu_t(z)$ is generated.

3.3 Efficiency

Following the same steps as in Section 2, we consider two separate margins. The first margin is the investment margin. In the planner's problem, the equation for this margin is (24):

$$\kappa=(1-\eta(\theta_t))q(\theta_t)\left[f_t^u\int_{\underline{z}_t} g(z')\mu(z')dz'+\int_{\underline{z}_t}^\infty\int_z^\infty f_t(z)g(z')\left(\mu_t(z')-\mu_t(z)\right)dz'dz\right].$$

In equilibrium, condition (17) represents the firm's incentive to invest in vacancy. Restating here,

$$\kappa = (1-\gamma)q(\theta_t)\left[f_t^u \int_{\underline{z}_t} g(z')S(z')dz + \int_{\underline{z}_t}^{\infty}\int_{z}^{\infty} f_t(z)g(z')\left[S_t(z') - \omega S_t(z)\right]dz'dz\right].$$

Let us compare equations (24) and (17). Suppose the Hosios condition (1) holds. If f_t^u and $f_t(z)$ are common and $\mu_t(z) = S_t(z)$, the equilibrium θ_t and the optimal θ_t coincide when the wage-setting protocol follows the sequential auction, that is, $\omega = 1$. If the wage-setting rules are different, the efficiency of the equilibrium θ is not guaranteed even in this situation. For example, with Nash bargaining, that is, $\omega = 0$, the value of equilibrium θ is too high compared to the social optimum. Intuitively, this result is because of the "worker-stealing" externality: the poaching firm does not internalize the loss of the poached firm.[5] With the sequential auction, the poaching firm pays for the loss (not to the poached firm, but to the worker) in the form of higher wages. Note that from (18) and (20) and the definitions of f^u and $f_t(z)$ (i.e., (6)), the steady-state values of f^u and $f(z)$ are the same when θ is the same. Thus, an important question for efficiency under the Hosios condition (and sequential auction in particular) is whether $\mu_t(z) = S_t(z)$ holds.

To see how $\mu_t(z)$ and $S_t(z)$ are determined, let us look at the valuation margin. On the valuation margin, we rewrite the equation for the social planner (22) as:

$$\begin{aligned}(r+\sigma)\mu_t(z) - \dot{\mu}_t(z) = {} & A_t z_t - h - p(\theta_t)\int_{\underline{z}_t} g(z')\mu_t(z')dz + \zeta p(\theta_t)\int_{z}^{\infty} g(z')(\mu_t(z') - \mu_t(z))dz' \\ & \underbrace{+(1-\eta(\theta))p(\theta)\left[f_t^u\int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{\tilde{z}}^{\infty} f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right]}_{\text{positive externality of not searching off the job}} \\ & \underbrace{-\zeta(1-\eta(\theta_t))p(\theta)\left[f_t^u\int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{\tilde{z}}^{\infty} f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right]}_{\text{negative externality of searching on the job}}.\end{aligned} \tag{25}$$

[5] This externality is analogous to the business-stealing externality in the economic growth literature. Gautier et al. (2010) emphasize a similar externality that arises in their model with the terminology of business-stealing externality.

The first term $A_t z - h$ on the right-hand side is the static gain from moving one unemployed worker to employment with the match quality z. The second term is the opportunity cost of not being able to do an off-the-job search. This cost is offset by the possibility of an on-the-job search, which is the last term in the first line. The second and third lines are externalities. The second line is the positive externality of not doing an off-the-job search, which raises the probability of a match for all the other workers. The third line is the negative externality of an on-the-job search, which lowers the probability of a match for all the other workers, as long as $\zeta > 0$.

For the equilibrium, we rewrite (15) to facilitate with the comparison with (25):

$$\begin{aligned}(r+\sigma)S_t(z) - \dot{S}_t(z) = A_t z - h - p(\theta_t)\int_{\underline{z}_t} g(z')S_t(z')dz' + \zeta p(\theta_t)\int_z^\infty g(z')\left(S_t(z') - S_t(z)\right)dz' \\ \underbrace{+(1-\gamma)p(\theta_t)\int_{\underline{z}_t}^\infty g(z')S_t(z')dz'}_{\text{cost of not searching off the job that is unaccounted for}} \\ \underbrace{-\zeta(1-\gamma)p(\theta_t)\int_z^\infty g(z')\left(S_t(z') - \omega S_t(z)\right)dz'}_{\text{benefit of searching on-the-job that is unaccounted for}}.\end{aligned} \tag{26}$$

The first line is analogous to the planner's solution (25). The second line is the opportunity cost of off-the-job search that is unaccounted for. By being employed, a worker loses the opportunity for off-the-job search, but that opportunity cost shows up only as a γ fraction, because the worker can receive only a γ fraction of the surplus due to Nash bargaining. Similarly, the third line is the benefit of on-the-job search that is unaccounted for.

Now let us compare equations (25) and (26) under the Hosios condition (1). The first three terms, which govern the static gain and the private opportunity costs, are identical. The final two terms (the second and the third line in the equations) are different. First, let us compare the second-line terms:

$$(1-\eta(\theta_t))p(\theta_t)\left[f_t^u\int_{\underline{z}_t}^\infty g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^\infty\int_{\tilde{z}}^\infty f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right] \tag{27}$$

and

$$(1-\gamma)p(\theta_t)\int_{\underline{z}_t}^\infty g(z')S_t(z')dz'. \tag{28}$$

The social planner's solution reflects that two externalities are associated with moving a worker out of unemployment. The first is that other unemployed workers have a higher probability of finding a match. The second is that employed workers have a higher probability of meeting with another job. The equilibrium outcome does not take into account either of these externalities. Instead, the corresponding term (28) expresses the worker's own opportunity cost that is unaccounted for.

Second, the final terms

$$-\zeta(1-\eta(\theta_t))p(\theta_t)\left[f_t^u\int_{\underline{z}_t}^{\infty}g(z')\mu_t(z')dz'+\int_{\underline{z}_t}^{\infty}\int_{\tilde{z}}^{\infty}f_t(\tilde{z})g(z')\left(\mu_t(z')-\mu_t(\tilde{z})\right)dz'd\tilde{z}\right] \tag{29}$$

and

$$-\zeta(1-\gamma)p(\theta_t)\int_{z}^{\infty}g(z')\left(S_t(z')-\omega S_t(z)\right)dz' \tag{30}$$

are analogous, corresponding to workers becoming employed. For the social planner, the externality from the on-the-job search partially offsets the externality in the previous term. For the equilibrium, once again, the gain from job-to-job transition is only partially accounted for.

In the valuation margin, $\mu_t(z)=S_t(z)$ is difficult to achieve. First, for the comparison of (27) and (28), one situation that would make these terms equal is $f_t^u=1$ and $f_t(z)=0$. This scenario, of course, does not occur unless $\zeta=0$ (i.e., no on-the-job search). Moreover, for the comparison of (29) and (30), $f_t^u=1$ and $f_t(z)=0$ do not yield the equivalence. For the terms (29) and (30), the opposite ($f_t^u=0$ and $f_t(z)=1$) brings the expression closer, but the difference remains. Here, a fundamental difference exists that is difficult to reconcile in this term. A worker's on-the-job search imposes externality to *all other* employed workers. The (unaccounted-for) opportunity cost in equilibrium is fundamentally about that particular match. Heterogeneity of z across workers is an essential feature of a job-ladder model, because no job-to-job transitions occur without such heterogeneity. This heterogeneity necessarily creates a disagreement between the externality to other matches and the opportunity cost within the own match.

In the above intuition, we highlight the importance of two factors that are impediments to efficiency. First, in equilibrium, moving a worker from unemployment to employment imposes externalities both to the unemployed workers and other employed workers. Second, because the employed workers are heterogeneous, offsetting the externalities to the other employed workers by proportionally reducing the value of equilibrium surplus that is counted as the opportunity cost is difficult.

To clarify these two factors further, in Appendix B, we analyze a model where (i) the matching functions for the unemployed and the employed are segmented, and (ii) for the employed, the matching functions for different values of z are segmented. We find that the equilibrium outcome is efficient with the Hosios condition and sequential auction ($\omega = 1$).[6] Both are needed for efficiency—just segmenting the unemployed market and the employed market is not sufficient. The results is analogous to the efficiency result in Menzio and Shi (2011), who show the efficiency of a job-ladder model with directed search. Our model features random search, and for the efficiency, in addition to the segmentation, the Hosios condition and sequential-auction mechanism are needed.

4 Analytical characterizations

The intuition that we highlighted in the previous section carries over to broader range of models with a similar structure. In this section, we further exploit the simplicity of the current model to derive further analytical results. All proofs are contained in Appendix C.

The main result is Proposition 1 below. It shows that, under the Hosios condition and a wide range of wage-setting mechanisms, the equilibrium vacancies are excessive compared with the social efficiency. This result underscores that the negative externalities an employed worker (especially the one with a high z) imposes on other workers (both unemployed workers and the other employed workers) are difficult to reconcile.

Proposition 1 is important particularly in the context of policy evaluations. It implies assuming the Hosios condition is not "neutral" in a job-ladder model: it automatically favors a policy that discourages vacancy posting. It is in contrast

[6] We thank Xincheng Qiu for suggesting this exercise.

to the model without an on-the-job search, where the Hosios condition implies no intervention is optimal absent other distortions. In the calibration of quantitative DMP literature, the Hosios condition is often assumed (e.g., Shimer, 2005). In the case of a job-ladder model, this calibration implies that policies that encourage or discourage vacancy posting give rise to inefficiency even without other distortions.

4.1 Market equilibrium

The decentralized equilibrium admits characterization by a system of two linear partial differential equations. Taking the derivative of (15) with respect to z, the match surplus solves

$$\begin{aligned}&[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))]\,\partial_z S_t(z)\\&\qquad\qquad\qquad=A_t+(1-\gamma)(1-\omega)\zeta p(\theta_t)g(z)S_t(z)+\partial_z\dot{S}_t(z),\end{aligned}$$

with a boundary condition $S_t(\underline{z}_t)=0$. The reservation match quality $\underline{z}_t$ satisfies

$$0=A\underline{z}_t-h-\gamma(1-\xi)p(\theta_t)\int_{\underline{z}_t}^{\infty}g(z')S_t(z')dz'.$$

The second partial differential equation is the evolution of employment distribution given by (18).

Evaluating these partial differential equations at the steady state, the steady-state equilibrium can be characterized by two equations with two unknowns, which are useful later.

Lemma 1 *The steady-state equilibrium market tightness and reservation match quality, $\{\theta,\underline{z}\}$, jointly solve*

$$0=A\underline{z}-h-A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z} \tag{31}$$

and

$$\begin{aligned}\kappa=A(1-\gamma)q(\theta)\Bigg[&\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz\\&+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg],\end{aligned} \tag{32}$$

where $\Gamma(\tilde{z};\theta) \equiv r + \sigma + (1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))$. *Moreover,* (31) *defines a weakly increasing relationship between* $\underline{z}$ *and* θ*, which we write as* $\underline{z}^R(\theta)$*, and* (32) *defines a strictly decreasing relationship between* $\underline{z}$ *and* θ*, which we write as* $\underline{z}^{FE}(\theta)$*.*

Given $\{\underline{z},\theta\}$, the rest of the equilibrium can be obtained as follows. The steady-state match surplus is given by

$$S(z) = A\int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z} \quad \text{for} \quad z \geq \underline{z},$$

the steady-state employment distribution is

$$N(z) = \frac{\left(G(z) - G(\underline{z})\right) p(\theta) u}{\sigma + (1-G(z))\zeta p(\theta)}, \tag{33}$$

and the steady state unemployment rate is

$$u = \frac{\sigma}{\sigma + (1-G(\underline{z}))p(\theta)}. \tag{34}$$

4.2 Social planner's problem

As in the decentralized equilibrium, the social value of a job satisfies the following partial differential equation, which we obtain by taking the derivative of (22) with respect to z:

$$\left[r + \sigma + \zeta p(\theta_t)\right] \partial_z \mu_t(z) = A_t + \partial_z \dot{\mu}_t(z),$$

with a boundary codnition $\mu_t(\underline{z}_t) = 0$. The reservation match quality $\underline{z}$ solves

$$0 = A_t \underline{z}_t - h - (1-\zeta)p(\theta_t)\int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \kappa\theta_t(1-\zeta).$$

Evaluating these equations at the steady state, the efficient steady-state allocation can be tractably characterized as follows.

Lemma 2 *The efficient steady-state market tightness and reservation match quality* $\{\theta^{SP}, \underline{z}^{SP}\}$ *jointly solve*

$$\begin{aligned} 0 = {} & A\underline{z}^{SP} - h \\ & - A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma\eta(\theta^{SP}) + (1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma + (1-G(\tilde{z}))\zeta p(\theta^{SP})} \frac{1-G(\tilde{z})}{r + \sigma + \zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z} \end{aligned} \tag{35}$$

and

$$\kappa = A(1-\eta(\theta))q(\theta^{SP}) \int_{\underline{z}^{SP}}^{\infty} \frac{\sigma}{\sigma + \zeta p(\theta^{SP})(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}. \tag{36}$$

Moreover, there exists $k < 0$ such that if $\eta'(\theta) > k$, (35) defines a weakly increasing relationship between $\underline{z}^{SP}$ and θ, which we write as $\underline{z}^{SP,R}(\theta)$, and (36) defines a strictly decreasing relationship between $\underline{z}^{SP}$ and θ, which we write as $\underline{z}^{FE,R}(\theta)$.

Given $\{\underline{z}^{SP}, \theta^{SP}\}$, the social value of a job can be obtained as

$$\mu(z) = A \int_{\underline{z}^{SP}}^{z} \frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}, \tag{37}$$

and the employment distribution is given by (33) and (34).

4.3 Results

The main result of this section shows that, with worker bargaining power lower than the matching-function elasticity with respect to vacancy, the vacancy creation in the market economy is always excessive relative to the efficient level in the presence of on-the-job search.

Proposition 1 *Suppose $\gamma \leq \eta(\theta^{SP})$ for all θ. Then, the steady-state equilibrium market tightness θ in the market equilibrium is higher than the socially efficient level when $\zeta > 0$.*

An immediate corollary is that the equilibrium vacancy creation is always excessive relative to the efficient level under the Hosios condition.

The logic underlying the proof can be graphically explained in Figure 1. The downward-sloping red solid line plots the relationship $\underline{z}^{FE}(\theta)$ in the decentralized equilibrium, derived from (32). The downward-sloping red dotted line plots the corresponding relationship for the planner's solution, denoted $\underline{z}^{FE,SP}(\theta)$, derived from (36). These lines are downward sloping because a larger $\underline{z}$ implies a less frequent match formation, and thus a lower return from posting vacancy. It can be shown that the solid line ($\underline{z}^{FE}(\theta)$) lies to the right of the solid line ($\underline{z}^{FE,SP}(\theta)$) as long as $\zeta > 0$. Both equations (32) and (36) represent the optimal vacancy-posting condition. For a given reservation match quality ($\underline{z}$), the value of a vacancy is higher in the decentralized economy than in the social optimum.

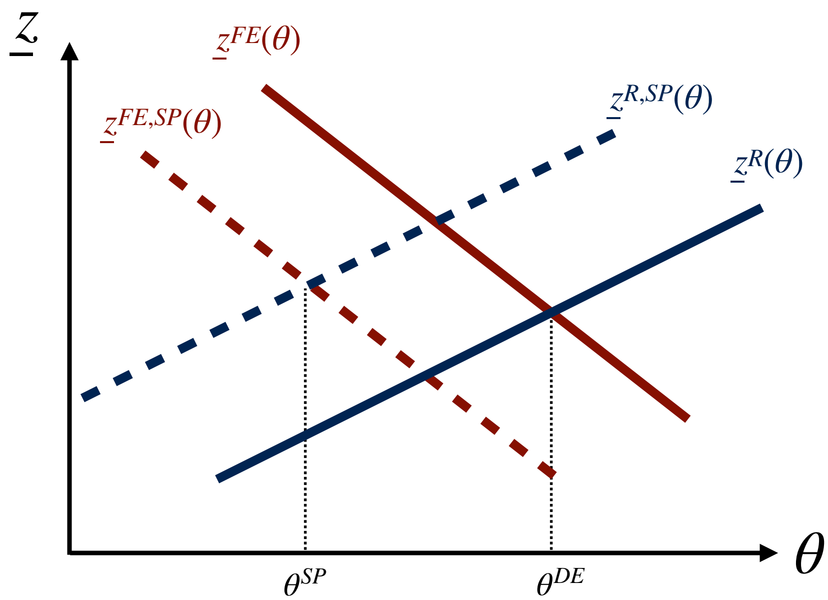

Figure 1: Inefficiency with on-the-job search

The upward-sloping blue solid line represents the equilibrium relationship $\underline{z}^R(\theta)$, derived from (31). The upward-sloping blue dotted line is the corresponding relationship for the social planner, $\underline{z}^{R,SP}(\theta)$, derived from (35). These lines are upward-sloping because a higher frequency of match allows the match formation to be more selective. Once again, the equilibrium relationship lies to the right of the social planner's relationship. The decentralized equilibrium values a job more than the planner, and therefore, they are more likely to form a match given θ. From the graph, we can see the outcome is a higher market tightness in the decentralized equilibrium than the efficient level, that is, $\theta^{DE} > \theta^{SP}$.

The natural next question is whether combinations of parameter values exist such that the decentralized equilibrium is efficient. Such situations can relatively easily be described in a special case with $\zeta = 1$. Under $\zeta = 1$ (the efficiency of on-the-job search is the same as unemployed), the reservation match quality in the decentralized equilibrium always coincides with the efficient level with $\underline{z} = h/A$. In such a situation, we can guarantee (the proof is omitted here) a worker bargaining parameter γ exists such that the efficiency is achieved. It can also be shown that, in such a case, γ is higher than what is prescribed by the Hosios condition.

Parameter	Description	Value	Target/Source
PANEL A. ASSIGNED PARAMETERS			
r	Discount rate	0.004	Annual interest rate 5%
η	Elasticity of matching function	0.5	Standard
γ	Worker bargaining power	0.5	Standard
ω	Offer matching	1.0	Cahuc et al. (2006)
h	Home production value	0.6	Normalization
A	Aggregate productivity	1.0	Normalization
PANEL B. CALIBRATED PARAMETERS			
σ	Separation rate	0.024	EU rate 2.4%
α	Pareto distribution shape parameter	5.2	UI elasticity of job-finding rate 0.37
z_{min}	Pareto distribution location parameter	2.5	Job-acceptance rate 49.4%
κ	Cost of vacancy creation	1.5	Market tightness 1
m	Matching efficiency parameter	0.83	Unemployment rate 5.5%
ζ	On-the-job search efficiency	0.25	EE rate 2.5%

Table 1: Parameter values

When $\zeta < 1$, two parameters (γ, ω) have to ensure the two equilibrium variables $(\underline{z}, \theta)$ are at the efficient level. Below, we numerically find such a combination of (γ, ω) in the quantitative experiment.

5 Quantitative exploration

In this section, we quantitatively explore the difference between the efficient allocation and the equilibrium outcome. We calibrate the economy in a standard manner; that is, we try to stay as close as possible to the existing macroeconomic literature. Knowing whether the difference between the efficient allocation and equilibrium outcome is small or large is of interest because the policy implications would be different. The magnitude of desired policy interventions, for example, depends on the magnitude of distortions in the economy.

5.1 Calibration

Table 1 summarizes our parameterization. We interpret one period as a month and set the discount rate to 5% annually, $r = 0.05/12$. The exogenous separation rate is set to 2.4% at monthly frequency, $\sigma = 0.024$. We assume the Cobb-Douglas matching function, $M(u, v) = mu^{\eta}v^{1-\eta}$ with $\eta = 0.5$, and the bargaining power is such that $\gamma = \eta$ (i.e., the Hosios condition), which are standard calibrations in the literature (e.g., Engbom, 2019). We measure units of vacancy so that $\theta \equiv v/u = 1$ in the steady state. We assume the productivity distribution is given by the Pareto distribution: $G(z) = 1 - (z/z_{min})^{-\alpha}$, which is consistent with a notion of a balanced growth path in Martellini and Menzio (2020). The flow value of unemployment is normalized to $h = 0.6$, and the aggregate productivity is normalized to $A = 1$.[7] We set the offer-matching parameter to $\omega = 1$, which corresponds to the sequential-auction protocol of Cahuc et al. (2006) and is widely used as a benchmark in the subsequent literature.

We then choose five parameters $\{m, \alpha, z_{min}, \kappa, \zeta\}$ to jointly match the following five moments: the steady-state unemployment rate of 5.5%; the monthly employment-to-employment transition rate of 2.5%; the job-acceptance rate of unemployed of 0.494, as reported in Faberman et al. (2022); and the partial-equilibrium elasticity of the job-finding rate with respect to the unemployment benefit, h, of 0.37, which corresponds to median estimates in the literature as surveyed by Schmieder and Von Wachter (2016). We compute the partial-equilibrium elasticity of the job-finding rate with respect to the unemployment benefit by simulating a model with a small change in h and computing the percentage change in the job-finding rate, holding the market tightness θ fixed.

Although all parameters are jointly calibrated, we provide a heuristic argument for how each moment identifies each parameter. First, the matching-efficiency parameter m is identified from the steady-state unemployment rate. The efficiency of on-the-job search, ζ, is inferred from the employment-to-employment transition

[7] With a Pareto match-quality distribution and perfect offer matching ($\omega = 1$), z_{min}, κ, and h are not separately identified, because any combination of (z_{min}, h, κ) with the same z_{min}/h and κ/h result in the same labor market allocation. Therefore, the choice of h merely amounts to normalization.

	Equilibrium	Efficient
Market tightness	1.00	0.78
Reservation match quality	2.865	2.889
Consumption	3.7725	3.7751
UE rate	0.412	0.349
EE rate	0.025	0.023
Unemployment rate	5.5%	6.4%

Table 2: Comparison between equilibrium vs. efficient allocation

Note: The table shows the aggregate variable in the decentralized equilibrium and in the efficient allocation.

rate. The lower bound of the productivity distribution, z_{min}, is identified from the job-acceptance rate of the unemployed. The cost of vacancy creation, κ, is chosen to ensure our normalization of $\theta = 1$. The elasticity of the job-finding rate with respect to the unemployment benefit identifies the Pareto-distribution shape parameter, α, because it controls the mass of workers near the reservation match quality.

5.2 Results: Decentralized equilibrium vs. efficient allocation

Now, we compare the outcomes of the decentralized equilibrium and the efficient allocation. Table 2 compares the aggregate variables. The equilibrium allocation has too many vacancies; θ is too high. As a result, the unemployment rate is too low and the aggregate productivity (the average z among the employed) is too high. The resulting consumption is lower in the equilibrium allocation, although the difference (about 0.07%) is small. The unemployment rate is 0.9 percentage points higher in the efficient allocation. The difference in the unemployment rate, therefore, is economically significant.

The discrepancy in the aggregate variables masks the underlying discrepancy in the distribution across job ladders. The top-left panel of Figure 2 compares the density function of z across the job ladder in the decentralized equilibrium with that in the planner's solution. The figure shows two features. First, because θ is larger in equilibrium, the distribution at the top is thicker for the market equilib-

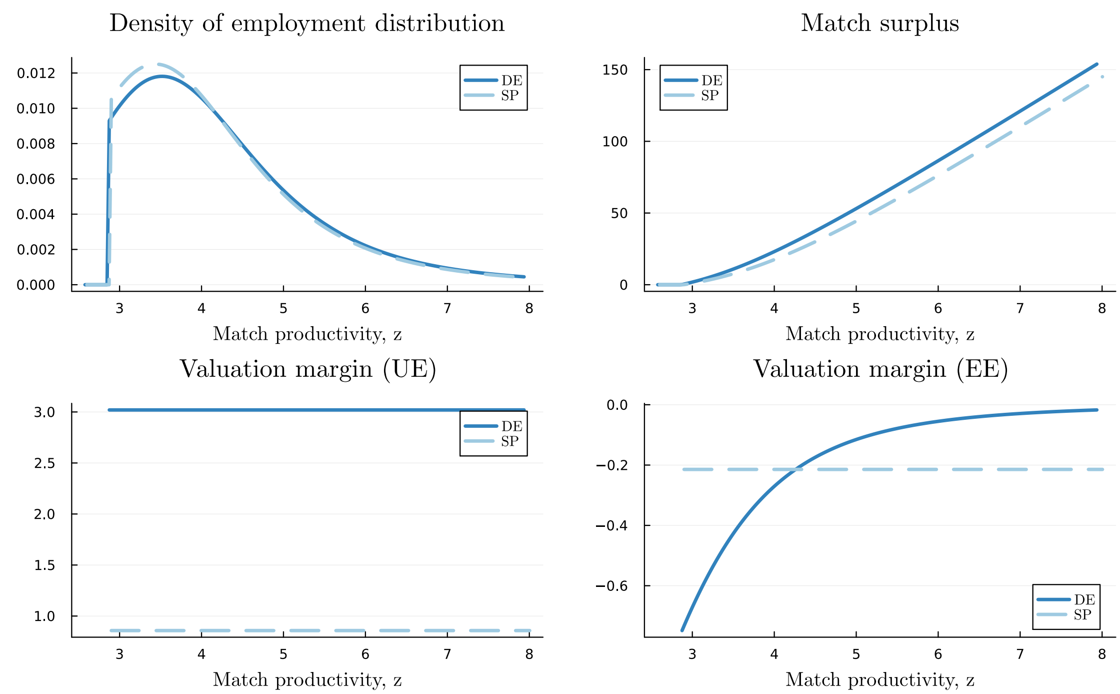

Figure 2: Employment distribution and match surplus: Equilibrium vs. planner

Note: The top-left panel compares the density of employment across the job ladder in the decentralized equilibrium and in the efficient allocation, $n(z)$. The top-right panel compares match surplus in the decentralized equilibrium, $S(z)$, and the planner's valuation of a match, $\mu(z)$. The bottom-left panel compares terms (27) and (28). The bottom-right panel compares terms (29) and (30).

rium. Second, at the bottom, the social planner is pickier in forming a match; thus, the bottom part of the distribution is more extended in the market equilibrium. All in all, the job-ladder in the market equilibrium features more dispersion in z. In the market equilibrium, too much inequality in earning exists.

The result of too many productive jobs in the decentralized equilibrium is in contrast to models with ex-ante job heterogeneity, such as Acemoglu (2001), where too few "good jobs" tend to exist in equilibrium compared with the efficient allocation.

In Section 3.3, we established that the valuation margin causes the discrepancy between the equilibrium allocation and the optimal allocation (in the case of the Hosios condition and sequential auction). To visualize the discrepancy, the top-right panel of Figure 2 compares the surplus in the decentralized equilibrium ($S(z)$; solid line in the left panel) with the planner's valuation ($\mu(z)$; dashed line in the left panel). Consistent with our earlier discussion, the planner systematically places a lower match value for each z.

To further investigate the difference between $S(z)$ and $\mu(z)$, in the bottom two panels in Figure 2, we plot the final two terms of the valuation margin, that is, term (27) versus term (28) and term (29) versus term (30). For (27) and (28) (the bottom-left panel), the equilibrium value is significantly larger. For (29) versus (30) (the bottom-right panel), note first that (30) is a function of z and (29) is independent of z. In equilibrium, a worker with high z has small job-to-job gains, and thus, the "missing private gain" due to the initial Nash bargaining is not large (in absolute value). By contrast, because of the random-search assumption, the externality one person imposes due to on-the-job search is the same regardless of z. For workers with large z, therefore, the effect of the negative externality they impose on others is significant.

In sum, the value of employment in equilibrium is excessively high for two reasons: (i) The negative externality an unemployed worker imposes on others is relatively small compared with the effect that in equilibrium, the value of unemployed search is undervalued; and (ii) for a high-z employed, the negative externality that they impose on others by their on-the-job search is large (in absolute value) relative to the effect that in equilibrium, on-the-job search for these workers is privately undervalued.

5.3 Steady-state comparative statics

This section conducts several comparative statics in terms of important parameters. We ask two questions. First, how does the existence of on-the-job search matter? We vary ζ from 0 (no on-the-job search) to 1 (on-the-job search is as efficient as off-the-job search) to examine this question. Second, how does the wage matching by the poaching firm matter? We have seen in Section 3.3 that when $\omega = 0$, the investment margin suffers from the worker-stealing externality, and the perfect offer matching (sequential auction) $\omega = 1$ alleviates this issue.

For the first question, Figure 3 varies the degree of on-the-job search ζ. The other parameters are kept constant at the values in the baseline. Because we impose the Hosios condition, the market equilibrium is efficient when $\zeta = 0$ (no

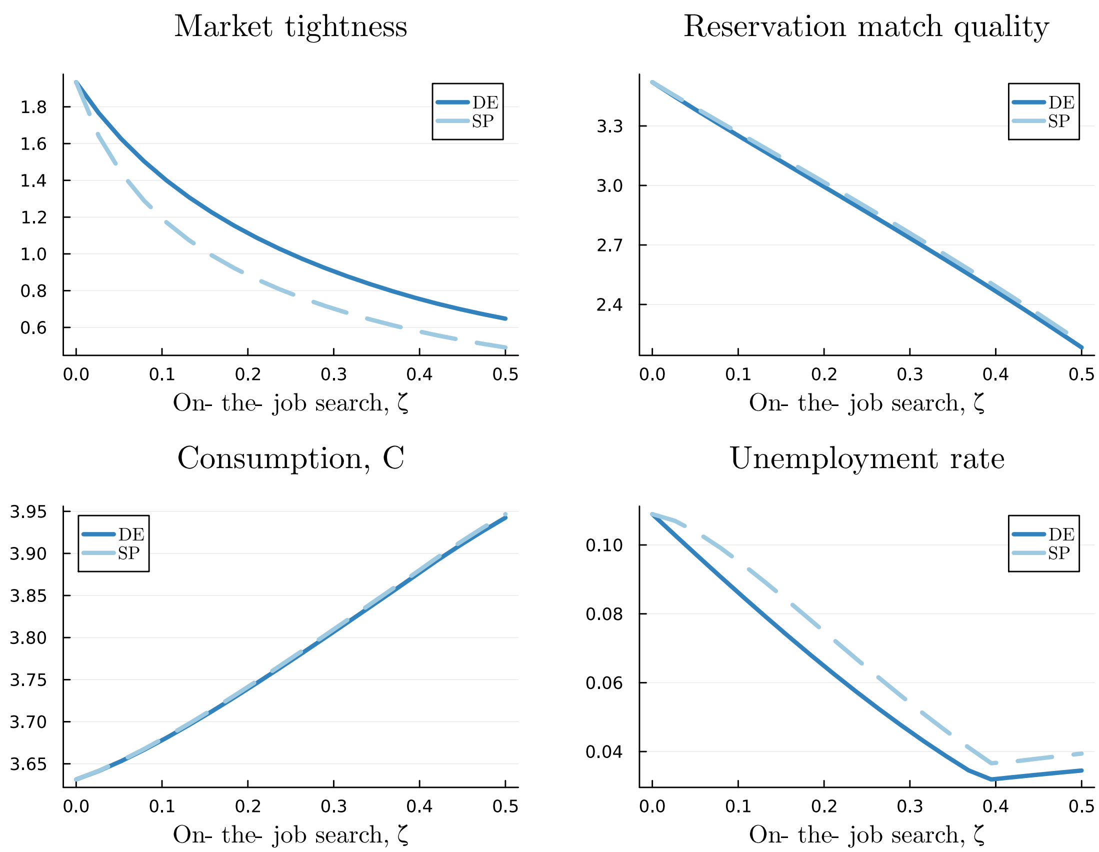

Figure 3: Varying the on-the-job search parameter ζ: equilibrium vs. planner

on-the-job search). What is striking here is that the labor market tightness in the market equilibrium significantly departs from the optimal outcome even with a small amount of on-the-job search; for example, $\zeta = 0.1$.

For the second question, Figure 4 varies the degree of offer matching ω. We have seen our benchmark, $\omega = 1$, achieves optimal investment in vacancy when valuation is correct. Therefore, it is not surprising that the consumption tends to be farther away from the efficient outcome when ω is lower. Although the labor market tightness increases with ω, the reservation match quality $\underline{z}$ also increases. The latter effect dominates in shaping the unemployment rate. The difference in the equilibrium unemployment rate between $\omega = 0$ and $\omega = 1$ is almost 2 percentage points, and thus, the worker-stealing externality is economically significant.

Reservation match quality increases with ω because the workers become choosier

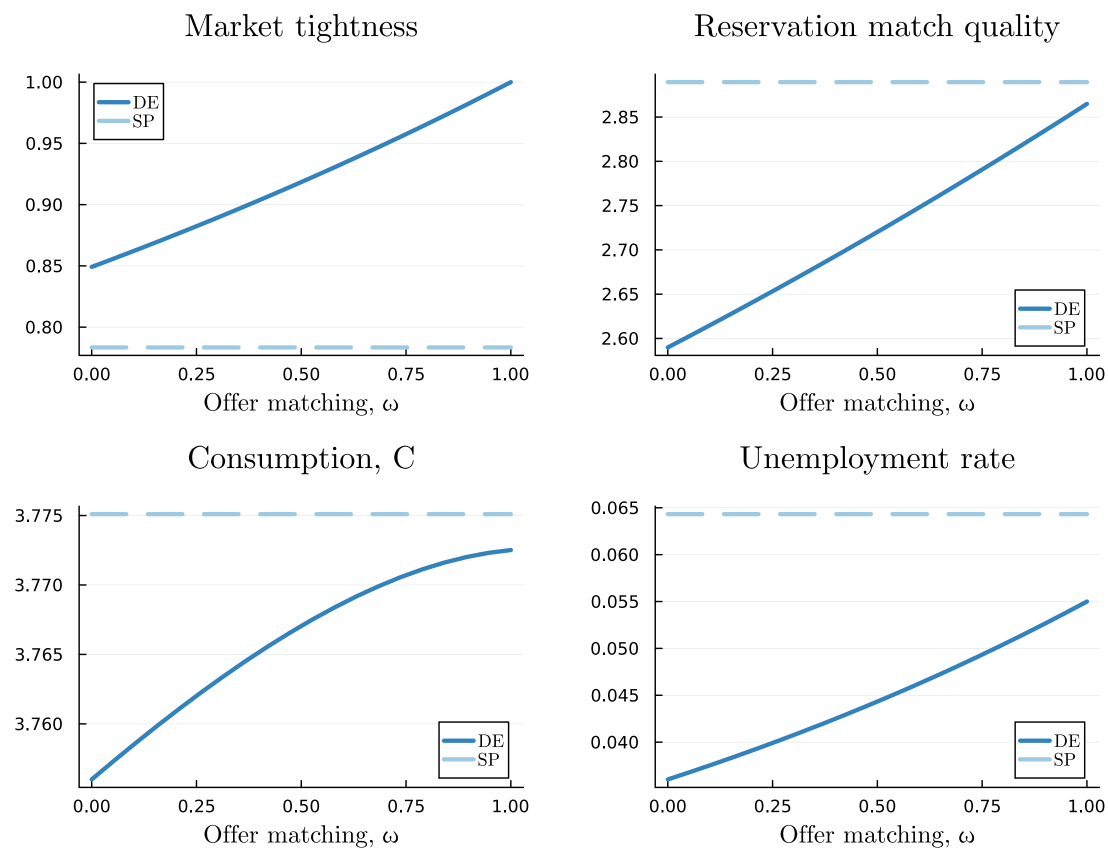

Figure 4: Varying the offer matching parameter ω: equilibrium vs. planner

with a higher θ. But why does θ increase? This result seems counterintuitive given that a higher ω implies a poaching firm has to pay a higher wage, discouraging the vacancy posting—θ should fall with ω for a *given* $S(z)$. However, $S(z)$ is not given. The valuation-margin equation (15) in fact reveals $S(z)$ increases with ω. This result holds because workers are forward looking: the workers' surplus (thus, the current match surplus) includes the future possibility of receiving a higher wage from the poaching firm. This increase in $S(z)$ (for given θ and $\underline{z}$) provides a higher motivation for vacancy posting. In our calibration, the latter effect dominates.

Finally, we ask whether a pair of bargaining parameters, (γ, ω), exists such that the steady-state allocation is efficient. The answer turns out to be yes. Table 3 shows the values of worker bargaining power, γ, and the offer-matching parameter, ω, that induce the same allocation (same θ and $\underline{z}$) in the steady state and

	Calibration	Efficient
Worker bargaining power, γ	0.50	0.57
Offer matching, ω	1.00	0.98

Table 3: Efficient Bargaining Parameters

Note: The table shows the bargaining parameters used in our calibration and the ones that achieve the efficient allocation.

contrast them with the calibration. We find that worker bargaining power γ that is higher than the Hosios condition and offer-matching parameter ω that is lower than the sequential auction achieves the efficient allocation.

5.4 Transition dynamics

So far, our focus has been on the steady state. We now consider the response of the economy to the productivity shock. This experiment is important in evaluating how the efficiency properties interact with various phases of the business cycle.

We consider a one-time unanticipated shock ("MIT shock") to the aggregate productivity, A_t, that increases by 1% at $t = 0$, which decays with rate ρ_A:

$$d \ln A_t = -\rho_A \ln(A_t/A)dt.$$

We set $\rho_A = 0.04$, which corresponds to the autocorrelation of 0.96 at the monthly frequency. We compare the response of the social planner's solution with that of the decentralized equilibrium, starting from the steady state described earlier. We solve the first-order approximation of transition dynamics around the steady state in sequence space, following the approach of Auclert et al. (2021). We describe the details of algorithms in Appendix D.

The top-left panel of Figure 5 shows the path of aggregate productivity. The top-right panel shows the unemployment rate goes down more in the decentralized equilibrium than in the planner's solution. Given the steady-state unemployment rate was already lower in the equilibrium than in the planner's solution, the positive productivity shock exacerbates the gap in the unemployment rate. The bottom two panels show this difference is driven by a larger initial rise in market tightness and a persistent larger fall in reservation match quality. Both of these

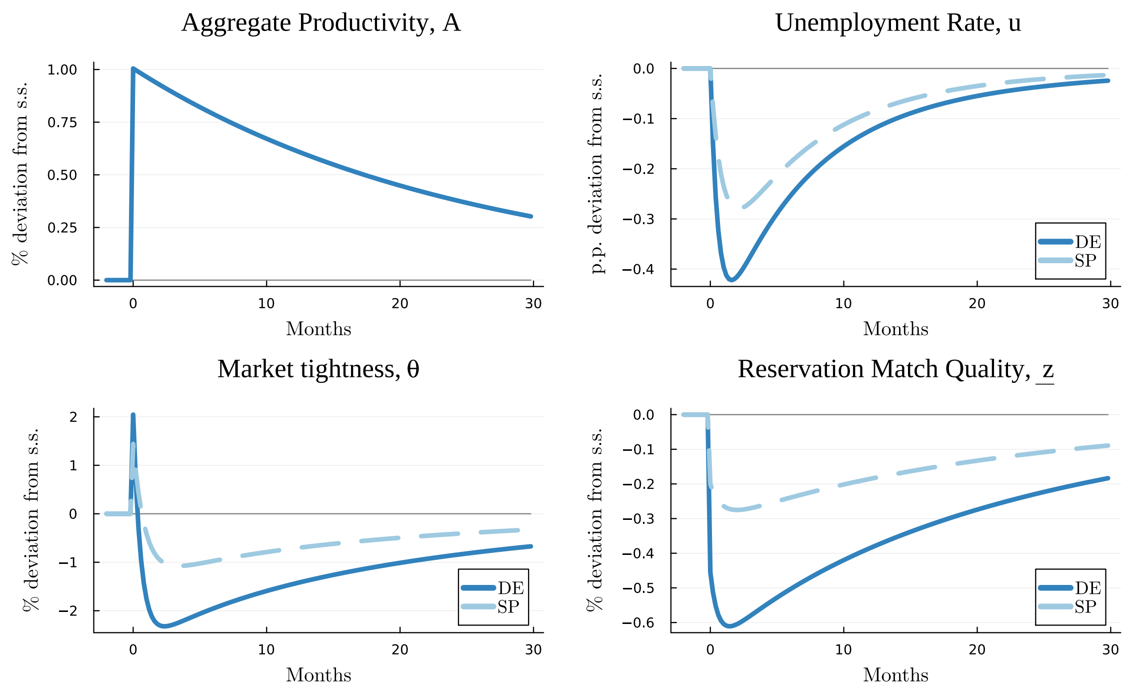

Figure 5: Response to productivity shock: Equilibrium and planner solution

Note: The figure plots the response of the decentralized equilibrium and the social planner's solution to a 1% initial increase in aggregate productivity at $t = 0$. The solid line represents the decentralized equilibrium, and the dashed line represents the social planner's solution.

effects are driven by the fact that a positive productivity shock induces the further over-valuation of the match in the equilibrium allocation relative to the planner's allocation. We make this point precise by analyzing how the wedge between the match surplus and the social value of a match changes in response to the changes in productivity below in Section 6.1.

6 Decentralization

In this section, we implement the efficient allocation through taxes. This exercise is of direct policy relevance, because knowing how much (and what kind of) taxes should be imposed and how they depend on the economic situation is useful. Expressing the magnitude of inefficiencies through "wedges," or implicit taxes, as in Chari et al. (2007), is also useful.

6.1 Optimal taxes

The efficient allocation is implementable in various ways. Here, we focus on two policy instruments, output tax and entry (vacancy-posting) tax. Suppose the government can impose an output tax $\tau(z)$ on a match so that the output net of tax is $Az(1 - \tau(z))$. Also, suppose the government imposes entry tax τ^e so that the vacancy-posting cost that the firms face is given by $(1 + \tau^e)\kappa$. The government finances these fiscal instruments through a lump-sum tax on workers. The equilibrium definition remains unchanged, where we replace Az with $Az(1 - \tau(z))$ and κ with $(1 + \tau^e)\kappa$.

The following lemma shows these two instruments not only implement efficient allocation but also ensure all jobs are valued the same as in the planner's solution.

Lemma 3 *There exist output taxes $\{\tau(z)\}$ and an entry tax τ^e such that the decentralized equilibrium implements efficient allocation with $S(z) = \mu(z)$ for all z.*

We further derive the expressions for optimal output taxes and the entry tax in Appendix C.4.

Although efficient allocation can be implemented in other ways, we focus on the implementation through output tax and entry tax for two reasons.[8] The first is theoretical clarity. As we discussed earlier, the inefficiency can be decomposed into two margins: the valuation margin and the investment margin. The two policy instruments correct the inefficiency in each margin. The output tax ensures the equilibrium match surplus coincides with the social value of a match. The entry tax ensures the equilibrium vacancy creation coincides with the efficient level, given the surplus is valued efficiently.

The second reason is the robustness with respect to other margins of inefficiency. For example, if the separation rate depends on the endogenous effort of the match, as in Balke and Lamadon (2022), for example, ensuring the match surplus coincides with the socially efficient level is necessary for the decentralized equilibrium to be efficient. In this case, output tax and entry tax still implement

[8] We show in Appendix E that a combination of the entry tax and the unemployment insurance also implements the efficient allocation.

the efficient allocation, whereas other policy instruments, such as unemployment insurance, do not. We make this point precise in Appendix F.

The following proposition characterizes the properties of the optimal output tax and entry tax.

Proposition 2 *Assume $\gamma \leq \eta(\theta^{SP})$ and $\zeta > 0$. Then, the following holds:*

1. *The steady-state optimal output tax is strictly positive, $\tau(z) > 0$ for all z, and satisfies $\lim_{z \to \infty} \tau(z) = 0$.*

2. *The entry tax is weakly positive $\tau^E \geq 0$. It is zero $\tau^E = 0$ when the Hosios condition holds, $\gamma = \eta(\theta)$, and offer-matching is perfect, $\omega = 1$.*

The proposition shows output tax and entry taxes are always positive as long as the workers' bargaining power is lower than the elasticity of the matching function and on-the-job searches occur. The entry tax is zero whenever the Hosios condition holds and the offer-matching is perfect, in which case, the investment margin is efficient. Even in such a case, the output tax is strictly positive, because the valuation margin is still inefficient. The tax *rate* is asymptotically zero as $z \to \infty$. In the top right panels of Figure 2, the gaps between the social planner's solution and the market equilibrium become asymptotically constant as $z \to \infty$. Thus, compared with z, the gap becomes smaller and smaller as z increases.

The next question we ask is whether these policies should systematically respond to changes in the underlying environment. To isolate the role of on-the-job search, we assume the Hosios condition holds: $\gamma = \eta(\theta)$ for all θ. Under this condition, the optimal taxes are zero without an on-the-job search. With on-the-job search, the output tax is given by

$$\tau(z) = \frac{1}{z} \int_{\underline{z}^{SP}}^{z} \left\{ \frac{\omega(1-\gamma)\zeta p(\theta^{SP})(1-G(\tilde{z}))}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} + \int_{\underline{z}^{SP}}^{\tilde{z}} \frac{(1-\gamma)(1-\omega)\zeta p(\theta^{SP}) g(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(z'))} dz' \right\} d\tilde{z},$$

which is strictly increasing in labor market tightness, θ^{SP}, and strictly decreasing in the reservation match quality, $\underline{z}^{SP}$. This result implies that, in response to booms in the labor market (where the labor market is tight and reservation match quality is low), the output tax should rise.

Proposition 3 *Suppose the Hosios condition holds, $\gamma = \eta(\theta)$ for all θ, and ζ is sufficiently close to 1. Then, the steady-state output tax, $\tau(z)$, is strictly increasing in the aggregate productivity A.*

What the above proposition illustrates is that the inefficiency along the valuation margin is more severe when the aggregate productivity is high. An increase in productivity exacerbates the overvaluation of jobs in the decentralized equilibrium relative to the planner. This overvaluation tends to lead to excessive vacancy creation and inefficiently low reservation match quality. This observation also explains our finding in Section 5.4 that the equilibrium unemployment rate exhibits excessive volatility. The response of the entry tax is generally ambiguous, but when the offer-matching is perfect, $\omega = 1$, it is invariant to the aggregate productivity.

6.2 Quantitative results

We illustrate the optimal output tax and entry tax that restore efficiency. Figure 6 plots the optimal steady-state output tax rate, $\tau(z)$, under our baseline parameterization. Because we assume the Hosios condition holds and offer matching is perfect, the optimal output tax is strictly positive and the optimal entry tax is zero, as shown in Proposition 2. Consistent with Proposition 2, the output tax rate decreases with z when z is sufficiently large.

The natural next question is how the policy should respond to the productivity shock. Figure 7 shows the optimal output and entry tax in response to the productivity shock. The optimal output tax increases sharply after the shock and reverts to the original level quickly. This result implies inefficiency along the valuation margin is larger in response to the positive productivity shock. By contrast, the entry tax does not respond. Once again, this result is due to our assumptions of the Hosios condition and perfect offer matching.

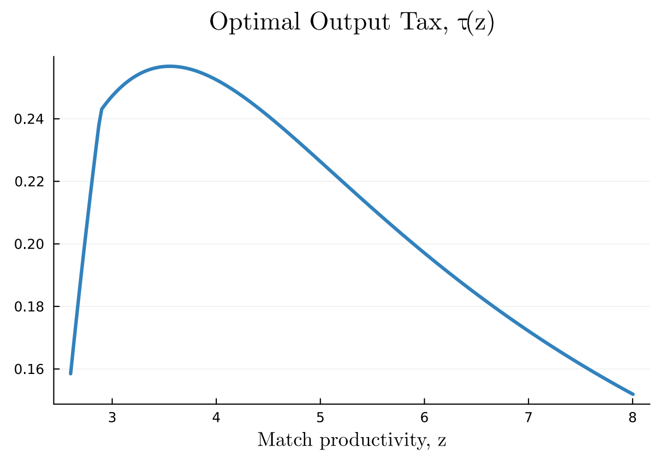

Figure 6: Optimal steady-state output tax, $\tau(z)$

Note: The figure plots the optimal steady-state output tax, $\tau(z)$, against match productivity, z.

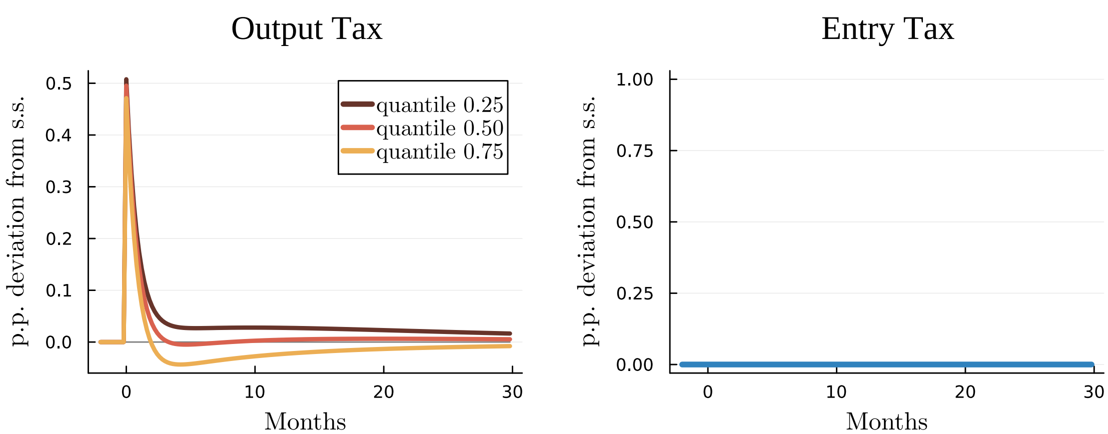

Figure 7: Optimal tax response to productivity shock

Note: The figure plots the response of the decentralized equilibrium and the social planner's solution to a 1% initial increase in aggregate productivity at $t = 0$, as in the left-top panel of Figure 5. The solid line represents the social planner's solution, and the dashed line represents the decentralized equilibrium.

7 Conclusion

This paper analyzes the efficiency of market equilibrium in a DMP-style job-ladder model. We identify two margins where inefficiencies can arise: the investment margin and the valuation margin. In the presence of an on-the-job search, the Hosios condition does not ensure the efficiency of market equilibrium. Even when we impose a wage-setting protocol that makes the investment margin undistorted, the valuation margin is still distorted. We show that, for a wide range of wage-setting protocols, too many vacancies are posted in market equilibrium under the Hosios condition. These results have important implications on the policy evaluations using the job-ladder models.

The quantitative exercise focuses on the situation where the economy is calibrated as in the standard macroeconomic literature, and the Hosios condition is satsified. It reveals that even with a small amount of on-the-job search, the optimal unemployment rate and the equilibrium unemployment rate can differ significantly. Thus, the inefficiencies highlighted in the earlier sections are quantitatively important. We also examine how the efficient allocations can be decentralized by output taxes and an entry tax.

The intuitions on inefficiencies would carry over to more complex models with various other factors. When we think about further complex models, an important factor is how the labor markets for different workers are segmented. The segmentation can be in different dimensions—here, we focused on the employment status (unemployed vs. employed) and the match quality, but the actual labor market can be segmented across various other dimensions, such as gender, race, education, experience, and geography. Our paper shows understanding the degree of segmentation in the economy is an important input in considering desirable policies under a frictional labor market.

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Online Appendix for "Efficiency in Job-Ladder Models"

A Pissarides (1985) model

In this Appendix, we spell out the entire Pissarides (1985) model and derive the Hosios condition by comparing the social planner's solution and the market outcome.

A.1 Social planner's problem

The social planner's problem is

$$\max_{n_t,\theta_t} \int_0^\infty e^{-rt}[zn_t + h(1-n_t) - \kappa\theta_t(1-n_t)]dt$$

subject to

$$\dot{n}_t = p(\theta_t)\left(1-n_t\right) - \sigma n_t.$$

The current-value Hamiltonian for this problem can be written as

$$H = zn_t + h\left(1-n_t\right) - \kappa\theta_t\left(1-n_t\right) + \mu_t\left(p(\theta_t)\left(1-n_t\right) - \sigma n_t\right).$$

Thus first-order conditions are

$$\kappa(1-n_t) = p'(\theta_t)\mu_t(1-n_t) \tag{38}$$

and

$$z - h + \kappa\theta_t - p(\theta_t)\mu_t - (r+\sigma)\mu_t + \dot{\mu}_t = 0. \tag{39}$$

From (38), noting $p'(\theta) = (1-\eta(\theta))q(\theta)$,

$$\kappa = (1-\eta(\theta))q(\theta)\mu_t.$$

This equation corresponds to (2) in the main text. From (39) in the steady state where $\dot{\mu}_t = 0$,

$$(r+\sigma)\mu_t = z - h - (p(\theta_t)\mu_t - \kappa\theta_t)$$

holds. This equation corresponds to (4) in the main text.

A.2 Market equilibrium

In the market equilibrium, the Hamilton-Jacobi-Bellman (HJB) equation for an unemployed worker is

$$rU_t = h + p(\theta_t)(W_t - U_t) + \dot{U}_t, \tag{40}$$

where U_t is the value of an unemployed worker and W_t is the value of employed worker. The HJB equation for an employed worker is

$$rW_t = w_t - \sigma(W_t - U_t) + \dot{W}_t, \tag{41}$$

where w_t is the wage at time t. On the firm side, the HJB equation for a matched job is

$$rJ_t = z - w_t - \sigma(J_t - V_t) + \dot{J}_t, \tag{42}$$

where J_t is the value of a matched job and V_t is the value of a vacancy. The value of vacancy satisfies

$$rV_t = -\kappa + q_t(J_t - V_t) + \dot{V}_t. \tag{43}$$

We assume anyone can post a vacancy (free entry), and therefore, the value of vacancy is driven down to zero in an equilibrium where the vacancy posting is strictly positive (we focus on such an equilibrium):

$$V_t = 0. \tag{44}$$

The wages are determined by Nash bargaining, which implies

$$W_t = \gamma S_t + U_t \tag{45}$$

and

$$J_t = (1-\gamma)S_t + V_t, \tag{46}$$

where $S_t \equiv W_t + J_t - U_t - V_t$ is the joint surplus from a match, and γ is workers' baragining power. Thus, an employed worker and a matched firm share the surplus with the fractions γ and $(1-\gamma)$, in addition to their outside options. Using (44) and (46) in (43), we obtain

$$\kappa = (1-\gamma)q(\theta_t)S_t.$$

This equation corresponds to (3) in the main text. Adding up (40), (41), and (42) and using (44) and (45) (also imposing the steady-state condition $\dot{S}_t = 0$), we obtain

$$(r+\sigma)S_t = z - h - p(\theta_t)\gamma S_t,$$

which corresponds to (5) in the main text.

B Segmented market

We modify our baseline model by assuming the labor market is segmented by workers' employment status as well as the match quality of the current jobs if workers are employed. We denote the market tightness that unemployed workers face as θ^u, and the market tightness that employed workers with match quality z face as $\theta^e(z)$. Free entry of firms is assumed in each of the segmented markets.

The value of being unemployed is given by

$$rU_t = h + p(\theta_t^u)\int g(z)\max\{W_t(z,U_t) - U_t, 0\}dz + \dot{U}_t.$$

The value function of workers employed with match quality z and an outside option $\bar{O} \in [U, U+\omega S(z)]$ is given by

$$\begin{aligned} rW_t(z,\bar{O}) = {} & w_t(z,\bar{O}) \\ & + \zeta p(\theta_t^e(z))\int_z^\infty g(z')\left(W_t(z', U_t + \omega S_t(z)) - W_t(z,\bar{O})\right)dz' \\ & + \zeta p(\theta_t^e(z))\int_0^z g(z')\left(W(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - W(z,\bar{O})\right)dz' \\ & + \sigma(U_t - W_t(z,\bar{O})) + \dot{W}_t(z,\bar{O}). \end{aligned}$$

The value of filled job with productivity z and outside option $\bar{O} \in [U, U+\omega S(z)]$ is given by

$$\begin{aligned} rJ_t(z,\bar{O}) = {} & A_t z - w_t(z,\bar{O}) \\ & + \zeta p(\theta_t^e(z))\int_z^\infty g(z')\left(V_t - J_t(z',\bar{O})\right)dz' \\ & + \zeta p(\theta_t^e(z))\int_0^z g(z')\left(J_t(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - J_t(z,\bar{O})\right)dz' \\ & + \sigma\left(V_t - J_t(z,\bar{O})\right) + \dot{J}_t(z,\bar{O}). \end{aligned}$$

The value of vacancy in the market for unemployed workers is given by

$$V^u = -\kappa + q(\theta_t^u) \int_0^\infty g(z) \max\{J_t(z, U) - V_t, 0\} dz + \dot{V}^u,$$

and the value of vacancy in the market for employed workers with match quality z is given by

$$V^e(z) = -\kappa + q(\theta_t^e(z)) \int_z^\infty f(z') \max\{J_t(z', U + \omega S_t(z)) - V_t, 0\} dz' + \dot{V}^e(z).$$

The free-entry condition is

$$V_t = \max\{V^u, \max_z V^e(z)\} = 0.$$

With Nash bargaining, the surplus from a match is

$$\begin{aligned}(r+\sigma)S_t(z) = A_t z - h \\ &+ \zeta p(\theta^e(z)) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right) dz' \\ &- p(\theta_t^u) \int_{\underline{z}_t}^\infty g(z') \gamma S_t(z') dz' + \dot{S}_t(z)\end{aligned} \tag{47}$$

for $z \geq \underline{z}_t$, where $\underline{z}_t$ satisfies

$$S(\underline{z}_t) = 0.$$

The matches with $z < \underline{z}_t$ are not formed.

The free-entry conditions are (assuming all markets have strictly positive vacancy posting)

$$\kappa = (1-\gamma) q(\theta_t^u) \int_{\underline{z}} g(z) S_t(z) dz \tag{48}$$

and

$$\kappa = (1-\gamma) q(\theta_t^e(z)) \int_z^\infty g(z') \left[S_t(z') - \omega S_t(z)\right] dz'. \tag{49}$$

Now we turn to the social planner's problem. The social planner's problem is to choose $\{\theta_t^u, \theta_t^e(z), n_t(z), \mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z, z'), \varsigma_t\}$ to maximize

$$\int_0^\infty e^{-rt} \left[\int_0^\infty A_t z n_t(z) dz + h\left(1 - \int_0^\infty n_t(z) dz\right) - \kappa\theta_t^u \left(1 - \int_0^\infty n_t(z) dz\right) - \kappa \int_0^\infty \theta_t^e(z) \zeta n_t(z) dz\right] dt$$

subject to

$$\dot{n}_t(z) = \left(1 - \int n_t(z')dz'\right) p(\theta_t^u) g(z) \mathbb{I}_t^{UE}(z) + \int_0^\infty p(\theta_t^e(z')) g(z) \mathbb{I}_t^{EE}(z', z) \zeta n_t(z') dz'$$

$$- \int_0^\infty p(\theta_t^e(z)) g(z') \mathbb{I}_t^{EE}(z, z') \zeta n_t(z) dz' - \sigma n_t(z) - \varsigma_t.$$

The current-value Hamiltonian for this problem is

$$\begin{aligned} H = & \int_0^\infty A_t z n_t(z) dz + h\left(1 - \int_0^\infty n_t(z) dz\right) - \kappa\theta_t^u \left(1 - \int_0^\infty n_t(z) dz\right) - \kappa \int_0^\infty \theta_t^e(z) \zeta n_t(z) dz \\ & + \int_0^\infty \mu_t(z) \Bigg[\left(1 - \int_0^\infty n_t(z') dz'\right) p(\theta_t^u) g(z) \mathbb{I}_t^{UE}(z) + \zeta \int_0^\infty p(\theta_t^e(z')) g(z) \mathbb{I}_t^{EE}(z', z) n_t(z') dz' \\ & \qquad - \zeta \int_0^\infty p(\theta_t(z)) g(z') n_t(z) \mathbb{I}_t^{EE}(z, z') dz' - \sigma n_t(z) - \varsigma_t \Bigg] dz, \end{aligned}$$

where $\mu_t(z)$ is the costate variable that represents the shadow value of the constraint (21). Thus, $\mu_t(z)$ is the shadow value of creating one unit of match with match quality z.

The optimality condition for $\{\mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z, z')\}$ is

$$\mathbb{I}_t^{UE}(z) = \begin{cases} 1 & \mu_t(z) > 0 \\ 0 & \mu_t(z) \leq 0 \end{cases},$$

and

$$\mathbb{I}_t^{EE}(z, z') = \begin{cases} 1 & \mu_t(z') > \mu_t(z) \\ 0 & \mu_t(z') \leq \mu_t(z) \end{cases}.$$

The optimality condition for endogenous separation ς_t implies $\mu_t(z) \geq 0$ for all z with $n_t(z) > 0$.

The first-order optimality condition on $n_t(z)$ is

$$\begin{aligned} (r + \sigma)\mu_t(z) = & A_t z - h - \int_{\underline{z}_t}^\infty p(\theta_t^u) g(z') \mu_t(z') dz' + \zeta p(\theta_t^e(z)) \int_z^\infty g(z')(\mu_t(z') - \mu_t(z)) dz' \\ & + \kappa\theta_t^u - \kappa\zeta\theta_t^e(z) + \dot{\mu}_t(z), \end{aligned} \tag{50}$$

where we have already imposed the fact that $\mu_t(z)$ is increasing in z. The reservation match quality $\underline{z}_t$ satisfies

$$\mu_t(\underline{z}_t) = 0.$$

The first-order optimality condition for θ_t^u is

$$\kappa = (1 - \eta(\theta_t^u))q(\theta_t^u) \int_{\underline{z}}^{\infty} \mu_t(z')g(z')dz' \tag{51}$$

The first-order condition with respect to $\theta_t^e(z)$ is

$$\begin{aligned} \kappa = {} & (1 - \eta(\theta_t^e(z)))q(\theta_t^e(z)) \int_0^{\infty} \mu_t(z')g(z')\mathbb{I}_t^{EE}(z, z')dz' \\ & - (1 - \eta(\theta_t^e(z)))q(\theta_t^e(z)) \int_0^{\infty} g(z')\mathbb{I}_t^{EE}(z, z')dz'\mu_t(z), \end{aligned} \tag{52}$$

which we can rewrite as

$$\kappa = (1 - \eta(\theta_t^e(z)))q(\theta_t^e(z)) \int_z^{\infty} g(z') \left[\mu_t(z') - \mu_t(z)\right] dz'.$$

Using free-entry condiitons, we can rewrite (50) as

$$(r+\sigma)\mu_t(z) = A_t z - h - \eta(\theta_t^u)p(\theta_t^u) \int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \zeta\eta(\theta_t^e(z))p(\theta_t^e(z)) \int_z^{\infty} g(z')(\mu_t(z') - \mu_t(z))dz'. \tag{53}$$

Comparing (47), (48), and (49) with (53), (51), and (52), we can see that the social planner's problem is equivalent to the decentralized equilibrium as long as Hosios condition holds, $\eta(\theta) = \gamma$ for all θ, and offer-matching is perefect, $\omega = 1$. We summarize the results as follows.

Proposition 4 *Consider the environment with segmented labor markets described above. The decentralized equilibrium is efficient if the Hosios condition holds, $\eta(\theta) = \gamma$ for all θ, and the offer-matching is perfect, $\omega = 1$.*

C Proofs

C.1 Proof of Lemma 1

In the steady state, the match surplus $S(z)$ satisfies

$$
\begin{aligned}
(r+\sigma)S(z) = {} & Az - h \\
& + \zeta p(\theta) \int_{z}^{\infty} g(z') \left((\omega-1)S(z) + \gamma \left[S(z') - \omega S(z) \right] \right) dz' \\
& - p(\theta) \int_{\underline{z}}^{\infty} g(z') \gamma S(z') dz'.
\end{aligned}
\tag{54}
$$

Taking a derivative with respect to z, we have

$$
\left[r + \sigma + (1-\omega(1-\gamma))\zeta p(\theta)(1-G(z)) \right] S'(z) + \zeta p(\theta) g(z)(\omega-1)(1-\gamma)S(z) = A.
$$

With a boundary condition $S(\underline{z}) = 0$, solving this ODE, we obtain

$$
S(z; \underline{z}, \theta) = \begin{cases} A \int_{\underline{z}}^{z} \frac{\left[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z)) \right]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\left[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z})) \right]^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z}. & \text{for} \quad z \geq \underline{z} \\ 0 & \text{for} \quad z < \underline{z}. \end{cases}
$$

Evaluating (54) at $z = \underline{z}$ yields

$$
0 = A\underline{z} - h - \gamma(1-\xi)p(\theta) \int_{\underline{z}}^{\infty} g(z') S(z'; \underline{z}, \theta) dz'.
$$

Applying integration by parts, we have

$$
\int_{\underline{z}}^{\infty} g(z') S(z'; \underline{z}, \theta) dz' = A \int_{\underline{z}}^{\infty} \frac{1}{\Gamma(\tilde{z}; \theta)} \left(\frac{\Gamma(\tilde{z}; \theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right) d\tilde{z},
$$

where

$$
\Gamma(z; \theta) \equiv r + \sigma + (1-\omega(1-\gamma))\zeta p(\theta)(1-G(z)).
$$

Plugging it back, we have

$$
\begin{aligned}
0 &= A\underline{z} - h - A\gamma(1-\zeta)p(\theta) \int_{\underline{z}}^{\infty} \frac{1}{\Gamma(\tilde{z}; \theta)} \left(\frac{\Gamma(\tilde{z}; \theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right) d\tilde{z} \\
&= H^R(\underline{z}, \theta).
\end{aligned}
$$

The solution to the above equation can be used to define the mapping $\underline{z}^R(\theta)$. Its derivative is

$$\frac{d\underline{z}^R(\theta)}{d\theta} = -\frac{\frac{\partial H^R(\underline{z},\theta)}{\partial\theta}}{\frac{\partial H^R(\underline{z},\theta)}{\partial\underline{z}}}.$$

Clearly, the denominator is positive, that is, $\partial H^R(\underline{z},\theta)/\partial\underline{z} > 0$. To sign the numerator,

$$\begin{aligned}
\frac{\partial H^R(\underline{z},\theta)}{\partial\theta} &= A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z} \\
-A\gamma(1-\zeta)p'(\theta)&\int_{\underline{z}}^{\infty}\frac{(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))}{\Gamma(\tilde{z};\theta)^2}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}\left(\frac{\gamma}{(1-\omega(1-\gamma))}\right)-G(\tilde{z})\right)d\tilde{z} \\
&\geq A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z} \\
&\quad -A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z} \\
&= 0.
\end{aligned}$$

Therefore, $z^R(\theta)$ is weakly increasing.

Let $P(z)$ denote the mass of workers employed with match quality below z or unemployed, where $P(\underline{z})$ corresponds to the unemployment rate. Its law of motion in the steady state is

$$0 = -(1-G(z))p(\theta)P(\underline{z}) - (P(z)-P(\underline{z}))(1-G(z))\zeta p(\theta) + \sigma(1-P(z)).$$

Solving for $P(z)$, we have

$$P(z) = \begin{cases} \frac{\sigma-(1-\zeta)(1-G(z))p(\theta)P(\underline{z})}{\sigma+(1-G(z))\zeta p(\theta)} & \text{for} \quad z \geq \underline{z} \\ \frac{\sigma}{p(\theta)(1-G(\underline{z}))+\sigma} & \text{for} \quad z < \underline{z}. \end{cases}$$

Define the probability that a vacancy meets with a worker already employed with match quality below z or unemployed is

$$\begin{aligned} F^{\ell}(z) &\equiv \frac{P(\underline{z}) + \zeta(P(z) - P(\underline{z}))}{P(\underline{z}) + \zeta(1 - P(\underline{z}))} \\ &= \begin{cases} \frac{\sigma}{\sigma + (1-G(z))\zeta p(\theta)} & \text{for} \quad z \geq \underline{z} \\ \frac{\sigma}{\sigma + (1-G(\underline{z}))\zeta p(\theta)} & \text{for} \quad z < \underline{z} \end{cases} \equiv F^{\ell}(z; \theta, \underline{z}). \end{aligned}$$

The associated density function is

$$f(z) = \frac{\sigma g(z) \zeta p(\theta)}{(\sigma + (1 - G(z))\zeta p(\theta))^2}.$$

Using $F^{\ell}(z; \theta, \underline{z})$, the free-entry condition in the steady state can be written as

$$\kappa = (1-\gamma) q(\theta) \int_0^{\infty} \int_{z'}^{\infty} g(z) \left[S(z; \underline{z}, \theta) - \omega S(z'; \underline{z}, \theta) \right] dz dF^{\ell}(z'; \theta, \underline{z}). \tag{55}$$

This defines a mapping $\underline{z}^{FE}(\theta)$.

Taking derivative,

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta} = - \frac{\frac{\partial}{\partial \theta} \left[q(\theta) \int_0^{\infty} \int_{z'}^{\infty} g(z) \left[S(z; \underline{z}, \theta) - \omega S(z'; \underline{z}, \theta) \right] dz dF^{\ell}(z'; \theta, \underline{z}) \right]}{q(\theta) \int_0^{\infty} \int_{z'}^{\infty} g(z) \left[\frac{\partial S(z; \underline{z}, \theta)}{\partial \underline{z}} - \omega \frac{\partial S(z'; \underline{z}, \theta)}{\partial \underline{z}} \right] dz dF^{\ell}(z'; \theta, \underline{z})}. \tag{56}$$

We would like to sign the above object. Note the denominator of (56) is negative because

$$\begin{aligned} \frac{\partial^2 S(z; \underline{z}, \theta)}{\partial z \partial \underline{z}} &= -A(1-\omega)(1-\gamma)\zeta p(\theta) g(z) \frac{\left[r + \sigma + (1 - \omega(1-\gamma))\zeta p(\theta)(1 - G(z)) \right]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))} - 1}}{\left[r + \sigma + (1 - \omega(1-\gamma))\zeta p(\theta)(1 - G(\underline{z})) \right]^{\frac{\gamma}{(1-\omega(1-\gamma))}}} \\ &\leq 0, \end{aligned}$$

and as a result,

$$\begin{aligned} q(\theta) \int_0^{\infty} \int_{z'}^{\infty} g(z) &\left[\frac{\partial S(z; \underline{z}, \theta)}{\partial \underline{z}} - \omega \frac{\partial S(z'; \underline{z}, \theta)}{\partial \underline{z}} \right] dz dF^{\ell}(z'; \theta, \underline{z}) \\ &\leq q(\theta) F^{\ell}(\underline{z}; \theta, \underline{z}) \int_0^{\infty} g(z) \frac{\partial S(z; \underline{z}, \theta)}{\partial \underline{z}} dz < 0. \end{aligned}$$

The numerator of (56) can be decomposed into (i) the effect through $q(\theta)$, (ii) the effect through $F^{\ell}(z'; \theta, \underline{z})$, and (iii) the effect through $S(z; \underline{z}, \theta)$. Note the first two are negative because $q'(\theta) < 0$ and $F^\ell(z';\theta',\underline{z})$ first-order stochastically dominates $F^\ell(z';\theta,\underline{z})$ for $\theta' > \theta$. To sign the third effect, note

$$\frac{\partial S(z;\underline{z},\theta)}{\partial\theta} = AB_1(z) + AB_2(z),$$

where

$$B_1(z) = -\eta(\theta)\frac{(1-\omega)(1-\gamma)\zeta p(\theta)(1-G(z))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))}S(z)$$

and

$$B_2(z) = -\eta(\theta)\int_{\underline{z}}^{z}\frac{\gamma\zeta p(\theta)(1-G(\tilde{z}))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))}\frac{\left[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))\right]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\left[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))\right]^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}.$$

Then,

$$\begin{aligned}\int_{z'}^{\infty}B_1(z)g(z)dz - \omega\int_{z'}^{\infty}B_1(z')g(z)dz &< -\eta(\theta)\frac{(1-\omega)(1-\gamma)\zeta p(\theta)(1-G(z'))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z'))} \\ &\times\left[\int_{z'}^{\infty}S(z)g(z)dz - \omega\int_{z'}^{\infty}S(z')g(z)dz\right] \\ &< 0\end{aligned}$$

and

$$\int_{z'}^{\infty}B_2(z)g(z)dz - \omega\int_{z'}^{\infty}B_2(z')g(z)dz < 0$$

because $B_2(z)$ is strictly decreasing. Therefore, the numerator is strictly negative. Putting all together, we have

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta} < 0.$$

Now, we seek to rewrite (55). We can rewrite the integral as follows:

$$\int_{\underline{z}}^{\infty}\int_{z'}^{\infty}g(z)\left[S(z)-\omega S(z')\right]dzdF^\ell(z') = \int_{\underline{z}}^{\infty}F^\ell(z)g(z)S(z)dz - \omega\int_{\underline{z}}^{\infty}(1-G(z))S(z)f(z)dz.$$

Using integration by parts,

$$\int_{\underline{z}}^{\infty} F^{\ell}(z)g(z)S(z)dz = -\left[F^{\ell}(z)(1-G(z))S(z)\right]_{\underline{z}}^{\infty} + A\int_{\underline{z}}^{\infty} f(z)(1-G(z))S(z)dz$$

$$+A\int_{\underline{z}}^{\infty} F^{\ell}(z)(1-G(z))S'(z)dz$$

$$= A\int_{\underline{z}}^{\infty} f(z)(1-G(z))\int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z}dz + A\int_{\underline{z}}^{\infty} F^{\ell}(z)(1-G(z))\frac{1}{\Gamma(z,\theta)}dz.$$

Substituting the above expressions back and using the expressions for $F^{\ell}(z)$ and $f(z)$, we have

$$\int_0^{\infty}\int_{z'}^{\infty} g(z)\left[S(z)-\omega S(z')\right]dzdF^{\ell}(z') = A\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz$$
$$+A(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz.$$

Now, (55) can be rewritten as follows:

$$\kappa = (1-\gamma)q(\theta)A\Bigg[\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz$$
$$+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg].$$

C.2 Proof of Lemma 2

Note $F^{\ell}(z;\theta)$ can be characterized in a same way as Lemma 1. We omit superscript SP for notational simplicity.

We start from equation (25) evaluated at the steady state:

$$(r+\sigma)\mu(z) = Az - h$$
$$-p(\theta)\int_{\underline{z}}^{\infty} g(z')\mu(z')dz + \zeta p(\theta)\int_{z}^{\infty} g(z')(\mu(z')-\mu(z))dz'$$
$$+(1-\zeta)(1-\eta(\theta))p(\theta)\left[\int_0^{\infty}\int_{z'}^{\infty} g(\tilde{z})\left(\mu(\tilde{z})-\mu(z')\right)d\tilde{z}dF^{\ell}(z';\theta,\underline{z})\right]. \quad (57)$$

Taking derivative with respect to z,

$$[r+\sigma+\zeta p(\theta)(1-G(z))]\,\mu'(z)=A.$$

Solving for $\mu(z)$ with a boundary condition $\mu(\underline{z})=0$ gives

$$\mu(z;\underline{z},\theta)=\begin{cases} A\int_{\underline{z}}^{z}\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}. & \text{for} \quad z\geq\underline{z} \\ 0 & \text{for} \quad z<\underline{z}\end{cases}$$

Evaluating (57) at $z=\underline{z}$ gives

$$\begin{aligned} 0=&\ A\underline{z}-h \\ &-(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}g(z')\mu(z';\underline{z},\theta)dz \qquad (58)\\ &+(1-\zeta)(1-\eta(\theta))p(\theta)\left[\int_{0}^{\infty}\int_{z'}^{\infty}g(\tilde{z})\left(\mu(\tilde{z};\underline{z},\theta)-\mu(z';\underline{z},\theta)\right)d\tilde{z}dF^{\ell}(z';\theta,\underline{z})\right].\end{aligned}$$

Now, we seek to simplify the above expression. We rewrite the second line using integration by parts:

$$\int_{\underline{z}}^{\infty}g(z')\mu(z';\underline{z},\theta)dz=A\int_{\underline{z}}^{\infty}(1-G(\tilde{z}))\frac{1}{r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}, \quad (59)$$

We rewrite the last term as

$$\begin{aligned} &\int_{0}^{\infty}\int_{z'}^{\infty}g(\tilde{z})\left(\mu(\tilde{z})-\mu(z';\underline{z},\theta)\right)d\tilde{z}dF^{\ell}(z') \\ =&\int_{\underline{z}}^{\infty}F^{\ell}(z')g(z')\mu(\tilde{z};\underline{z},\theta)d\tilde{z}-\int_{\underline{z}}^{\infty}(1-G(z'))\mu(z';\underline{z},\theta)f(z')dz' \\ =&\ A\int_{\underline{z}}^{\infty}F^{\ell}(\tilde{z})(1-G(\tilde{z}))\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\ =&\ A\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}, \qquad (60)\end{aligned}$$

where the second line changes the order of integration, and the third line uses integration by parts. Plugging (59) and (60) into (58), we obtain

$$0=A\underline{z}-h-A(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}$$

$$\equiv H^{R}(\underline{z},\theta).$$

The solution to the above equation gives a mapping $\underline{z}^R(\theta)$, and its derivative is given by

$$\frac{d\underline{z}^R(\theta)}{d\theta} = -\frac{\frac{\partial H^R(\underline{z},\theta)}{\partial \theta}}{\frac{\partial H^R(\underline{z},\theta)}{\partial \underline{z}}}.$$

Clearly, the denominator is positive, that is, $\frac{\partial H^R(\underline{z},\theta)}{\partial \underline{z}} > 0$. To sign the numerator,

$$\begin{aligned}
\frac{\partial H^R(\underline{z},\theta)}{\partial \theta} = A(1-\zeta)p(\theta) \int_{\underline{z}}^{\infty} & \frac{1-G(\tilde{z})}{(\sigma + (1-G(\tilde{z}))\zeta p(\theta))^2 (r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))^2} \\
& \times \Big[\sigma\eta(\theta) \left(\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)\right) \left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right) \\
& + \sigma\eta'(\theta) \left(\sigma + (1-G(\tilde{z}))\zeta p(\theta)\right) \left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right) \\
& + (1-G(\tilde{z}))\zeta\eta(\theta)p(\theta) \left(\sigma + (1-G(\tilde{z}))\zeta p(\theta)\right) \left(r+\sigma(1-\eta(\theta))\right) \Big] d\tilde{z}.
\end{aligned}$$

This is weakly positive if $\eta'(\theta) \geq k^R$, where

$$\begin{aligned}
k^R \equiv \max_{\tilde{z}} - \frac{\eta(\theta)\left(\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)\right)}{\left(\sigma + (1-G(\tilde{z}))\zeta p(\theta)\right)} - \frac{(1-G(\tilde{z}))\zeta\eta(\theta)p(\theta)(r+\sigma(1-\eta(\theta)))}{\sigma\left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right)} \\
< 0.
\end{aligned}$$

Therefore, $\underline{z}^R(\theta)$ is weakly increasing if $\eta'(\theta) \geq k^R$.

We can rewrite (24) to obtain:

$$\kappa = (1-\eta(\theta))q(\theta) \left[\int_0^{\infty} \int_{z'}^{\infty} g(\tilde{z}) \left(\mu(\tilde{z};\underline{z},\theta) - \mu(z';\underline{z},\theta)\right) d\tilde{z} dF^{\ell}(z';\theta,\underline{z}) \right].$$

Substituting (60), we have

$$\begin{aligned}
\kappa &= A(1-\eta(\theta))q(\theta) \int_{\underline{z}}^{\infty} \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} d\tilde{z} \\
&\equiv H^{FE}(\underline{z},\theta).
\end{aligned}$$

This defines a mapping $\underline{z}^{FE}(\theta)$. To sign this,

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta} = -\frac{\frac{\partial H^{FE}(\underline{z},\theta)}{\partial\theta}}{\frac{\partial H^{FE}(\underline{z},\theta)}{\partial\underline{z}}}.$$

Clearly, $\frac{\partial H^{FE}(\underline{z},\theta)}{\partial\underline{z}} < 0$, and the denominator is negative. To sign the numerator,

$$\begin{aligned}
&\frac{\partial H^{FE}(\underline{z},\theta)}{\partial\theta} \\
&= -A\eta'(\theta)q(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\
&\quad + (1-\eta(\theta))q'(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\
&\quad - (1-\eta(\theta))q(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(\tilde{z}))\zeta p'(\theta)(1-G(\tilde{z}))}{(\sigma+\zeta p(\theta)(1-G(\tilde{z})))}\frac{1}{(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))} \\
&\qquad\quad \times\left(\frac{1}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}+\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right)d\tilde{z}.
\end{aligned}$$

This is strictly negative if $\eta'(\theta) > k^{FE}$, where

$$\begin{aligned}
k^{FE} \equiv& -(1-\eta(\theta))\eta(\theta) \\
&- \frac{1}{\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}}\times \\
&(1-\eta(\theta))\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(\tilde{z}))\zeta p'(\theta)(1-G(\tilde{z}))}{(\sigma+\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))} \\
&\quad\times\left(\frac{1}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}+\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right)d\tilde{z} \\
&< 0.
\end{aligned}$$

Therefore, if $\eta'(\theta) > k^{FE}$, $z^{FE}(\theta)$ is strictly decreasing. Setting $k \equiv \max\{k^{FE}, k^R\} < 0$ completes the proof.

C.3 Proof of Proposition 1

We first show $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$ for all θ. Let

$$H^R(\underline{z},\theta) \equiv A\underline{z} - h - A\gamma(1-\xi)p(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z})\right)d\tilde{z}$$

and

$$H^{R,SP}(\underline{z},\theta) \equiv A\underline{z} - h - A(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty} \frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} d\tilde{z}.$$

Because $\underline{z}^R(\theta)$ and $\underline{z}^{R,SP}(\theta)$ are the solutions to $H^R(\underline{z},\theta)=0$, and $H^{R,SP}(\underline{z},\theta)=0$ and both H^R and $H^{R,SP}$ are increasing in $\underline{z}$, $H^R(\underline{z},\theta) > H^{R,SP}(\underline{z},\theta)$ for all $(\underline{z},\theta)$ is sufficient for $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$. Note that $H^R(\underline{z},\theta) > H^{R,SP}(\underline{z},\theta)$ holds if

$$\frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{\Gamma(\tilde{z};\theta)} \left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right) \geq 0$$

for all $\tilde{z}$.

To show the above inequality,

$$\frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{\Gamma(\tilde{z};\theta)} \left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right)$$

$$\geq (1-G(\tilde{z})) \left(\frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)} \frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z}))} \right)$$

$$= (1-G(\tilde{z})) \frac{(\eta(\theta)-\gamma)\sigma\left(r+\sigma+(1-G(\tilde{z}))\zeta p(\theta)\right) + (1-G(\tilde{z}))\zeta p(\theta)(r+\sigma)(1-\gamma)}{(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z})))},$$

which is strictly positive whenever $\eta(\theta) \geq \gamma$. Therefore $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$ for all θ.

Next, we show $\underline{z}^{FE}(\theta) \geq \underline{z}^{FE,SP}(\theta)$ for all θ. They are given by the solutions to $H^{FE}(\underline{z},\theta)=0$ and $H^{FE,SP}(\underline{z},\theta)=0$, respectively, where

$$H^{FE}(\underline{z},\theta) = A(1-\gamma)q(\theta)\Bigg[\int_{\underline{z}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)} \frac{1}{\Gamma(z,\theta)} dz$$

$$+ (1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2} \int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z}dz\Bigg] - \kappa$$

and

$$H^{FE,SP}(\underline{z},\theta) = A(1-\eta(\theta))q(\theta)\int_{\underline{z}}^{\infty} \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} d\tilde{z} - \kappa.$$

Because $H^{FE}(\underline{z},\theta)$ and $H^{FE,SP}(\underline{z},\theta)$ are both increasing in $\underline{z}$, showing $H^{FE}(\underline{z},\theta) > H^{FE,SP}(\underline{z},\theta)$ for all $(\underline{z},\theta)$ is sufficient for $\underline{z}^{FE}(\theta) \geq \underline{z}^{FE,SP}(\theta)$. To show this,

$$\begin{aligned}
& H^{FE}(\underline{z},\theta) - H^{FE,SP}(\underline{z},\theta) \\
= & A(1-\gamma)q(\theta)\left[\int_{\underline{z}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)} \left(\frac{1}{\Gamma(z,\theta)} - \frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right) dz \right. \\
& \left. + (1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2} \int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z} dz \right] \\
& + A(\eta(\theta)-\gamma)q(\theta)\int_{\underline{z}}^{\infty} \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} d\tilde{z}.
\end{aligned}$$

The right-hand side is strictly positive whenever $\eta(\theta) \geq \gamma$.

Putting all together, we have shown θ is strictly higher in the decentralized equilibrium than the efficient level, as in Figure 1.

C.4 Proof of Lemma 3

We state the following lemma, which characterizes the taxes that implement the efficient allocation. The following lemma immediately implies Lemma 3.

Lemma 4 *The efficient steady-state allocation can be implemented by a combination of output tax $\{\tau(z)\}$ and entry tax τ^e that satisfy*

$$\begin{aligned}
\tau(z) = & \frac{1}{z}\int_{\underline{z}^{SP}}^{z} \left\{ \frac{\omega(1-\gamma)\zeta p(\theta^{SP})(1-G(\tilde{z}))}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} + \zeta p(\theta^{SP})g(\tilde{z})\int_{\underline{z}^{SP}}^{\tilde{z}} \frac{(1-\gamma)(1-\omega)}{r+\sigma+\zeta p(\theta^{SP})(1-G(z'))} dz' \right\} d\tilde{z} \\
& + \frac{1}{z}(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{(1-G(\tilde{z}))}{r+\sigma+\gamma\zeta p(\theta^{SP})(1-G(\tilde{z}))} \left\{ \frac{(\eta(\theta^{SP})-\gamma)\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} \right\} d\tilde{z}
\end{aligned}$$

and

$$1+\tau^e = \frac{1-\gamma}{1-\eta(\theta^{SP})}\left\{1+(1-\omega)\frac{\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma g(z)\zeta p(\theta^{SP})(1-G(z'))}{(\sigma+(1-G(z'))\zeta p(\theta^{SP}))^2}\int_{\underline{z}^{SP}}^{z'}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}dz'}{\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma(1-G(z'))}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}}\right\}.$$

In such an equilibrium, $S(z) = \mu(z)$ for all z. Moreover, there exists $\bar{\gamma} \geq \eta(\theta^{SP})$ such that for $\gamma < \bar{\gamma}$, the output tax and the entry tax are both positive, $\tau(z) \geq 0$ for all z and $\tau^e \geq 0$.

Proof. In the decentralized equilibrium with taxes, the match surplus $S_t(z)$ solves

$$\begin{aligned}(r+\sigma)S_t(z) = {} & A_t z(1-\tau_t(z)) - h \\ & + \zeta p(\theta_t)\int_z^{\infty} g(z')\left((\omega-1)S(z)+\gamma\left[S_t(z')-\omega S_t(z)\right]\right)dz' \\ & - p(\theta_t)\int_{\underline{z}_t}^{\infty} g(z')\gamma S_t(z')dz' + \dot{S}_t(z). \end{aligned} \tag{61}$$

Evaluating this equation at the steady state and taking the derivative with respect to z, we have

$$\begin{aligned}\left[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))\right]S'(z) + \zeta p(\theta)g(z)(\omega-1)(1-\gamma)S(z) \\ = A(1-\tau(z)) - \tau'(z)Az.\end{aligned}$$

To ensure $S(z) = \mu(z)$ for all z, the output taxes have to satisfy

$$A\tau(z) + Az\tau'(z) = \omega(1-\gamma)\zeta p(\theta)(1-G(z))\mu'(z) + (1-\gamma)(1-\omega)\zeta p(\theta)g(z)\mu(z).$$

Using (37), we obtain the candidate solutions to the above ODE:

$$\begin{aligned} z\tau(z) = \int_{\underline{z}^{SP}}^{z}\Bigg\{ & \frac{\omega(1-\gamma)\zeta p(\theta^{SP})(1-G(\tilde{z}))}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} \\ & + (1-\gamma)(1-\omega)\zeta p(\theta^{SP})g(\tilde{z})\int_{\underline{z}^{SP}}^{\tilde{z}}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(z'))}dz'\Bigg\}d\tilde{z} + C, \end{aligned} \tag{62}$$

where C is a constant determined so that $S(\underline{z}^{SP}) = \mu(\underline{z}^{SP}) = 0$. At $\underline{z}^{SP}$, the output taxes solve

$$0 = A\underline{z}^{SP} - Az\tau(\underline{z}^{SP}) - h - \gamma(1-\xi)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} g(z')\mu(z')dz'.$$

Using (58), we can solve for $\tau(\underline{z}^{SP})$ as follows:

$$\underline{z}^{SP}\tau(\underline{z}^{SP}) = (1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{(1-G(\tilde{z}))}{r+\sigma+\gamma\zeta p(\theta^{SP})(1-G(\tilde{z}))}\left\{\frac{(\eta(\theta^{SP})-\gamma)\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\right\}.$$

Plugging $C = \underline{z}^{SP}\tau(\underline{z}^{SP})$ into (62), we obtain the output taxes that ensure $S(z) = \mu(z)$ for all z:

$$\begin{aligned}
z\tau(z) = &\int_{\underline{z}^{SP}}^{z}\left\{\frac{\omega(1-\gamma)\zeta p(\theta^{SP})(1-G(\tilde{z}))}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\right.\\
&\left.+(1-\gamma)(1-\omega)\zeta p(\theta^{SP})g(\tilde{z})\int_{\underline{z}^{SP}}^{\tilde{z}}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(z'))}dz'\right\}d\tilde{z}\\
&+(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{(1-G(\tilde{z}))}{r+\sigma+\gamma\zeta p(\theta^{SP})(1-G(\tilde{z}))}\left\{\frac{(\eta(\theta^{SP})-\gamma)\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\right\}.
\end{aligned}$$

Now, we look for the entry tax that ensures the efficient level of vacancy creation. The free-entry condition with taxes, after plugging in $S(z) = \mu(z)$, is given by

$$\kappa(1+\tau_t^e) = (1-\gamma)q(\theta_t)\left[f_t^u\int_{\underline{z}_t} g(z)S_t(z)dz + \int_0^{\infty}\int_{z'}^{\infty} f_t(z')g(z)\left[S_t(z)-\omega S_t(z')\right]dz'dz\right]. \tag{63}$$

The efficient level of vacancy creation solves (24). Using the fact that the output taxes ensure $S_t(z) = \mu_t(z)$ and imposing a steady state in (63), for the vacancy creation to coincide with (24), the entry tax τ^e needs to be set so that

$$1+\tau^e = \frac{(1-\gamma)\left[f^u\int_{\underline{z}^{SP}} g(z)\mu(z)dz + \int_0^{\infty}\int_{z'}^{\infty} f(z')g(z)\left[\mu(z)-\omega\mu(z')\right]dz'dz\right]}{(1-\eta(\theta^{SP}))\left[f^u\int_{\underline{z}^{SP}} g(z)\mu(z)dz + \int_0^{\infty}\int_{z'}^{\infty} f(z')g(z)\left[\mu(z)-\mu(z')\right]dz'dz\right]}. \tag{64}$$

Note

$$\begin{aligned}
&\int_0^{\infty}\int_{z'}^{\infty} g(\tilde{z})\left(\mu(\tilde{z})-\omega\mu(z';\underline{z},\theta)\right)d\tilde{z}dF^{\ell}(z')\\
&= A\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z'))}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}\\
&\quad+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z'))}{(\sigma+(1-G(z'))\zeta p(\theta))^2}A\int_{\underline{z}^{SP}}^{z'}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}dz'.
\end{aligned}$$

Plug the above expression into (64) to obtain

$$1+\tau^e = \frac{1-\gamma}{1-\eta(\theta^{SP})}\left\{1+(1-\omega)\frac{\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma g(z)\zeta p(\theta^{SP})(1-G(z'))}{(\sigma+(1-G(z'))\zeta p(\theta^{SP}))^2}\int_{\underline{z}^{SP}}^{z'}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}dz'}{\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma(1-G(z'))}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\frac{1}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}}\right\}.$$

■

C.5 Proof of Proposition 2

From the expression in Lemma 4, it is immediate to see $\tau(z) > 0$ when $\gamma \leq \eta(\theta^{SP})$. Moreover, the second line of the expression for $\tau(z)$ is strictly decreasing in z and goes to zero as $z \to \infty$. It is sufficient to show the first line of the expression goes to zero as $z \to \infty$. Applying L'Hôpital's rule,

$$\begin{aligned}\lim_{z\to\infty}\tau(z) &= \lim_{z\to\infty}\zeta p(\theta^{SP})g(z)\int_{\underline{z}^{SP}}^{z}\frac{(1-\gamma)(1-\omega)}{r+\sigma+\zeta p(\theta^{SP})(1-G(z'))}dz' \\ &\leq \lim_{z\to\infty}\zeta p(\theta^{SP})g(z)\frac{(1-\gamma)(1-\omega)}{r+\sigma}(z-\underline{z}^{SP}) \\ &= 0,\end{aligned}$$

where the last equality follows from the assumption that the mean of z is finite, and thus, $\lim_{z\to\infty} g(z)z = 0$.

C.6 Proof of Proposition 3

Because the output tax, $\tau(z)$, is strictly increasing in θ^{SP} and strictly decreasing in $\underline{z}^{SP}$, it is sufficient to show θ^{SP} is increasing and $\underline{z}^{SP}$ is decreasing in A. When ζ is sufficiently close to 1, (35) implies $\underline{z}^{SP}$ is indeed decreasing in A. To show θ^{SP} is increasing in A, we look at the right-hand side of (36), which is increasing in A, decreasing in θ, and decreasing in $\underline{z}^{SP}$ under the Hosios condition $\eta(\theta) = \gamma$ for all θ. Therefore, θ^{SP} is increasing in A. The second part of the statement follows from the expression in Lemma 4.

D Computational algorithms for transition dynamics

We describe the computational algorithms for the transition dynamics. Throughout, we focus on one-time unanticipated shocks starting from the steady state. We describe the transition dynamics for the decentralized equilibrium, and we allow for the presence of taxes. The transition dynamics for the planner's problem can be obtained in the same manner. The transition dynamics for the decentralized equilibrium with taxes are characterized by $\{S_t(z), \underline{z}_t, \theta_t, N_t(z), u_t\}$, which jointly solve (16), (18), (20), (61), and (63).

We solve the first-order approximation of the transition dynamics around the steady state in a sequence space, following the approach of Auclert et al. (2021). We discretize time with the time interval Δ, and the truncated horizon of the transition dynamics with finite period T.

We first solve the joint match surplus backward in time using (61), starting from the terminal condition $dS_T(z) = 0$, in response to small changes in aggregate productivity and market tightness at the terminal period, $\{dA_T, d\theta_T\}$. This gives the following derivatives:

$$\frac{dS_{T-s}(z)}{dA_T}, \quad \frac{dS_{T-s}(z)}{d\theta_T}.$$

We then represent the first-order response of the match surplus as a function of an arbitrary sequence of $\{A_s, \theta_s\}_s$ as follows:

$$d\boldsymbol{S}(z) = \mathcal{J}^{S,\theta}(z) d\boldsymbol{\theta} + \mathcal{J}^{S,A}(z) d\boldsymbol{A}, \tag{65}$$

where $d\boldsymbol{S}(z) \equiv [dS_t(z)]_t, d\boldsymbol{\theta} \equiv [d\theta_t]_t$, and $d\boldsymbol{A} \equiv [dA_t]_t$ are all $T/\Delta \times 1$ vectors, and $\mathcal{J}^{S,X}(z)$ is a $T/\Delta \times T/\Delta$ Jacobian matrix of match surplus with productivity z, $\boldsymbol{S}(z)$ with respect to $\boldsymbol{X}$. Each element of Jacobian can be obtained from a single backward iteration mentioned above as

$$\mathcal{J}^{S,X}_{t,t+s}(z) = \begin{cases} \frac{dS_{T-s}(z)}{dX_T} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},$$

for $X = \theta$ and A, which is the core insight of Auclert et al. (2021).

Second, we represent the response of reservation match quality as

$$
\begin{aligned}
d\underline{\boldsymbol{z}} &= \frac{1}{S'(\underline{z})} d\boldsymbol{S}(\underline{z}) \\
&= \frac{1}{S'(\underline{z})} \left[\mathcal{J}^{S,\theta}(\underline{z}) d\boldsymbol{\theta} + \mathcal{J}^{S,\theta}(\underline{z}) d\boldsymbol{A} \right]
\end{aligned} \tag{66}
$$

which follows from total differentiation of $S_t(\underline{z}_t) = 0$.

The third step is to compute the derivatives of the employment distribution with respect to the aggregate productivity, market tightness, and reservation match quality at time 0 to obtain the following objects:

$$
\frac{\partial u_t}{\partial A_0}, \quad \frac{\partial u_t}{\partial \theta_0}, \quad \frac{\partial u_t}{\partial \underline{z}_0}, \quad \frac{\partial N_t(z)}{\partial A_0}, \quad \frac{\partial N_t(z)}{\partial \theta_0}, \quad \frac{\partial N_t(z)}{\partial \underline{z}_0}.
$$

The first-order response of the distribution as a function of an arbitrary sequence of $\{A_s, \theta_s\}_s$ can be expressed as follows.

$$
d\boldsymbol{u} = \mathcal{J}^{u,\theta} d\boldsymbol{\theta} + \mathcal{J}^{u,A} d\boldsymbol{A} + \mathcal{J}^{u,\underline{z}} d\underline{\boldsymbol{z}}, \tag{67}
$$

$$
d\boldsymbol{N}(z) = \mathcal{J}^{N,\theta}(z) d\boldsymbol{\theta} + \mathcal{J}^{N,A}(z) d\boldsymbol{A} + \mathcal{J}^{N,\underline{z}}(z) d\underline{\boldsymbol{z}}, \tag{68}
$$

where the Jacobian matrix can be obtained as

$$
\mathcal{J}^{u,X}_{t+s,t} = \begin{cases} \frac{du_s}{dX_0} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},
$$

$$
\mathcal{J}^{N,X}_{t+s,t}(z) = \begin{cases} \frac{dN_s(z)}{dX_0} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},
$$

for $X = \theta$, A, and $\underline{z}$.

Because the free-entry condition, (63), is a function of $\{u_t, S_t(z), N_t(z), \theta_t, \underline{z}_t\}$, we can write it as $\tilde{\mathcal{H}}^{FE}(u_t, S_t(z), N_t(z), \theta_t, \underline{z}_t) = \kappa$. The linearized free-entry condition is

$$
\frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t} du_t + \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial \theta_t} d\theta_t + \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial \underline{z}_t} d\underline{z}_t + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')} dS_t(z') dz' + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial N_t(z')} dN_t(z') dz' = 0.
$$

Substituting (65), (67), and (68) into the above equation, we can write the linearized free entry condition as,

$$
\mathcal{H}^{FE,\theta} d\boldsymbol{\theta} + \mathcal{H}^{FE,A} d\boldsymbol{A} + \mathcal{H}^{FE,\underline{z}} d\underline{\boldsymbol{z}} = 0, \tag{69}
$$

where $\mathcal{H}^{FE,\theta}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t}\mathcal{J}^{u,\theta}_{t,s}$, $\mathcal{H}^{FE,A}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t}\mathcal{J}^{u,A}_{t,s} + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')}\mathcal{J}^{S,A}_{t,s}(z')dz'$, and $\mathcal{H}^{FE,\underline{z}}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t}\mathcal{J}^{u,\underline{z}}_{t,s} + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')}\mathcal{J}^{S,\underline{z}}_{t,s}(z')dz' + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial N_t(z')}\mathcal{J}^{N,\underline{z}}_{t,s}(z')dz'$.

Stacking (66) and (69), the first-order response of $d\underline{\boldsymbol{z}}, d\boldsymbol{\theta}$ solves

$$\begin{bmatrix} \mathcal{H}^{FE,\theta} & \mathcal{H}^{FE,\underline{z}} \\ \frac{1}{S'(\underline{z})}\mathcal{J}^{S,\theta}(\underline{z}) & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} d\boldsymbol{\theta} \\ d\underline{\boldsymbol{z}} \end{bmatrix} = - \begin{bmatrix} \mathcal{H}^{FE,A} \\ \frac{1}{S'(\underline{z})}\mathcal{J}^{S,A}(\underline{z}) \end{bmatrix} d\boldsymbol{A},$$

where $\boldsymbol{I}$ is $T/\Delta \times T/\Delta$ identity matrix. We can solve for $d\boldsymbol{\theta}, d\underline{\boldsymbol{z}}$ by inverting the matrix on the left-hand side. Given $d\boldsymbol{\theta}, d\underline{\boldsymbol{z}}$, the rest of the objects can be obtained using (65), (67), and (68).

E Implementation of efficient allocation through unemployment insurance and entry tax

We consider an alternative implementation of efficient allocation through unemployment insurance and entry tax. Unemployment insurance provides b units of consumption goods for unemployed workers so that the flow value of being unemployed is now given by $h + b$. The entry tax is given by τ^e and the cost of entry inclusive of tax is $(1 + \tau^e)\kappa$. These two policy instruments are sufficient to implement efficient allocation.

Proposition 5 *There exists a pair of unemployment insurance b and entry tax τ^e that implement the efficient allocation in the steady state. Moreover, there exists $\bar{\gamma} \geq \eta(\theta)$ such that for $\gamma < \bar{\gamma}$, the unemployment insurance and the entry tax are both positive, $b > 0$ and $\tau^e > 0$.*

Proof. In the steady state of the decentralized equilibrium with the above policy instruments, $(\underline{z}, \theta)$ solve

$$0 = A\underline{z} - h + b - A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty} \frac{1}{\Gamma(\tilde{z};\theta)} \left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right) d\tilde{z}$$

and

$$\kappa(1+\tau^e) = A(1-\gamma)q(\theta)\Bigg[\int_{\underline{z}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)} \frac{1}{\Gamma(z,\theta)} dz$$

$$+ (1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2} \int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z}dz\Bigg].$$

We compare the above two conditions with those for the efficient allocation:

$$0 = A\underline{z}^{SP} - h - A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma\eta(\theta^{SP}) + (1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma+(1-G(\tilde{z}))\zeta p(\theta^{SP})} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}$$

and

$$\kappa = A(1-\eta(\theta))q(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}.$$

Therefore, we can obtain the equivalence by setting

$$b = A\gamma(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{1}{\Gamma(\tilde{z};\theta^{SP})} \left(\frac{\Gamma(\tilde{z};\theta^{SP})^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z}) \right) d\tilde{z}$$

$$- A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma\eta(\theta^{SP}) + (1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma+(1-G(\tilde{z}))\zeta p(\theta^{SP})} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}$$

and

$$1+\tau^e = (1-\gamma)\Bigg[\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta^{SP})} \frac{1}{\Gamma(z,\theta^{SP})} dz$$

$$+ (1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta^{SP})(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta^{SP}))^2} \int_{\underline{z}}^{z} \frac{\Gamma(z,\theta^{SP})^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta^{SP})^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z}dz\Bigg]$$

$$\times \frac{1}{(1-\eta(\theta^{SP}))\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}}.$$

The proof of Proposition 1 already shows that when $\gamma \leq \eta(\theta^{SP})$,

$$b \geq 0, \quad \tau^e \geq 0.$$

By continuity in γ, there exists $\bar{\gamma} > \eta(\theta^{SP})$ such that for all $\gamma \leq \bar{\gamma}$, $h \geq 0$, and $\tau^e > 0$ hold. ■

Because the steady-state allocation can be essentially summarized by the two aggregates, $(\underline{z}, \theta)$, it should not be surprising that the two instruments can decentralize the efficient allocation. The second part of the proposition shows the optimal policy features a positive entry tax and positive unemployment insurance as long as worker bargaining power is not too high.

F Extension with endogenous effort

Consider an extension of the baseline model with taxes to an environment with endogenous effort choice that determines the separation rate. In particular, we assume workers can choose the level of effort, e, which determines the separation rate of a match, $\sigma(e)$. The separation rate is decreasing and concave in effort. The unit cost of effort is ι. The equilibrium joint match surplus in this environment is given by

$$\begin{aligned} rS_t(z) = \max_e\, & A_t z(1-\tau_t(z)) - h - \iota e - \sigma(e)S_t(z) \\ & + \zeta p(\theta) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right) dz' \\ & - p(\theta_t) \int_{\underline{z}_t}^\infty g(z')\gamma S_t(z') dz' + \dot{S}_t(z). \end{aligned}$$

The optimal choice of effort of the job with match quality z satisfies

$$\iota = \sigma'(e) S_t(z).$$

We denote the solution to the above expression as $e_t(z)$. The rest of the equilibrium conditions are the same as the baseline model except that the separation rate σ is replaced with $\sigma(e(z))$ for each match quality z.

Likewise, the valuation of the job in the planner's solution is given by

$$
\begin{aligned}
r\mu_t(z) = \max_e\, & A_t z - h - \iota e - \sigma(e)\mu_t(z) \\
& - \int_{\underline{z}_t}^{\infty} p(\theta_t) g(z')\mu_t(z')dz' + \zeta p(\theta_t) \int_{z}^{\infty} g(z')(\mu_t(z') - \mu_t(z))dz' \\
& + \kappa\theta_t(1-\zeta) + \dot{\mu}_t(z).
\end{aligned}
$$

The optimal choice of effort of the job with match quality z satisfies

$$\iota = \sigma'(e)\mu_t(z).$$

We denote the solution to the above expression as $e_t^{SP}(z)$.

The output tax $\tau(z)$ that ensures $\mu_t(z) = S_t(z)$ will also ensure the effort choices are efficient $e_t(z) = e_t^{SP}$. By contrast, because unemployment insurance or entry tax alone does not ensure $\mu_t(z) = S_t(z)$, the effort choices will not be efficient.